Some Solutions for Math 480 Week #1

Page 18 #25: Prove that there are infinitely many pairs of integers $x,\ y$ satisfying $x+y=100$ and $\gcd(x,y)$=5.

Write $y=100-x$ and $x=5a$. If neither 2 nor 5 divide a, then the only prime divisor of $100-x=100-5a=2^2 5^2-5a$ and $x=5a$ is 5, and $25 \nmid 100-x$. This makes $\gcd(x,100-x)$=5. Certainly there are infinitely many possibilities for a; for instance, a could be any prime not equal to 2 or 5.

#44: Prove that any positive integer can be written uniquely as a sum of the form $2^{j_0}+2^{j_1}+\cdots+2^{j_m}$, where $0 \le j_0 < j_1 < \cdots < j_m$.

The crucial observation is the following fact (look in your calculus text if you lack faith): for any non-negative integer z,

$$1+2+2^2+\cdots+2^z=2^{z+1}-1<2^{z+1}.$$

Using this, one can proceed by induction on n. The case $n=1$ is certainly true (let $m=0=j_0$). Now suppose $n>1$ and let z be the largest integer such that $2^{z+1} \le n$. Note that $z \ge 0$ since $n \ge 2$. If $n=2^{z+1}$ we're done. Otherwise, $n-2^{z+1}>0$, and by the induction hypothesis

$$n-2^{z+1}=2^{j_0}+\cdots+2^{j_m}$$

with $0 \le j_0 < \cdots < j_m$. By construction, $j_m \le z$, so that

$$n=2^{j_0}+\cdots+2^{j_m}+2^{z+1}.$$

So what about UNIQUENESS? Well, this can by proven by induction on n as well. Suppose first that $n=1=2^0$. If $2^0=2^{j_0}+\cdots+2^{j_m} \ge 2^{j_m}$, then certainly $j_m=0=m$. Now suppose $n>1$ and

$$n=2^{j_0}+\cdots+2^{j_m}=2^{k_0}+\cdots+2^{k_l}$$

with $0 \le j_0 < \cdots < j_m$ and $0 \le k_0 < \cdots < k_l$. Without loss of generality, $j_m \le k_l$. If $j_m < k_l$, then by what was shown above (the crucial observation),

$$n=2^{j_0}+\cdots+2^{j_m} \le 1+2+\cdots+2^{j_m} < 2^{k_l} \le n.$$

Thus $j_m = k_l$. So now look at $n - 2^{j_m} = n - 2^{k_l} < n$. By the induction hypothesis, $m - 1 = l - 1$ and $j_i = k_i$ for $i = 0, \ldots, m-1$. This proves the uniqueness of the expansion.

Page 29, #31: Prove that no polynomial $f(x)$ of positive degree with integral coefficients can represent a prime for every positive integer x.

The hint in the back of the book was pretty good, but you were supposed to prove the assertions in that hint. Suppose $f(x)$ is a polynomial of positive degree with integral coefficients. If $f(1)$ is not a prime, we're done. Otherwise, $f(1) = p$ for some prime p. By the binomial theorem, for a given integer k and any non-negative integer i, $(1 + kp)^i = 1 + k'p$ for some integer k' (depending on k and i). Thus, for a given integer k, $f(1 + kp) = f(1) + k''p = (1 + k'')p$ for some integer k''. Now this will be a prime if and only if $k'' = 0$, so that $f(1 + kp) = p$. But the equation $f(x) = p$ can have at most $\deg(f)$ solutions since $f(x)$ is not a constant. Therefore, $k'' = 0$ for only finitely many k, so that $f(1 + kp)$ is a prime for only finitely many k.