1. Find the linearization of $f(x) = \sqrt[3]{27+x}$ at $x = 0$. Use this to approximate the cube root of 27.1.

Setting $u = 27 + x$ and using the chain rule,

$$\begin{aligned} \frac{df}{dx} &= \frac{du^{1/3}}{du}\frac{d(27+x)}{dx} \\ &= (1/3)u^{-2/3}\left(\frac{d27}{dx} + \frac{dx}{dx}\right) \\ &= (1/3)(27+x)^{-2/3}(0+1). \end{aligned}$$

Thus, $f'(0) = (1/3)27^{-2/3} = 1/27$. Also, $f(0) = 27^{1/3} = 3$. The linearization at $x = 0$ is

$$L(x) = f'(0)(x-0) + f(0) = (1/27)(x-0) + 3.$$

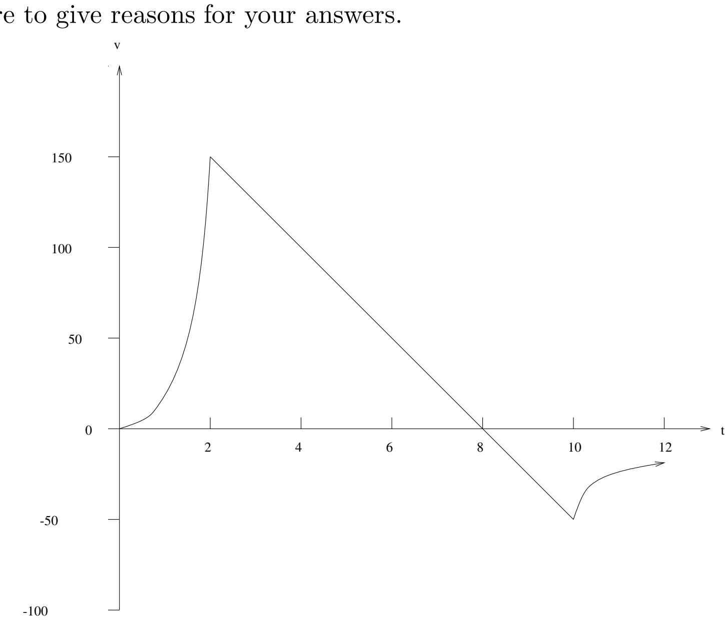
Using the linearization to approximate the function, we have

$$\sqrt[3]{27.1} = f(.1) \approx L(.1) = (1/27)(1/10) + 3.$$

2. Below is the graph of the velocity v (in feet/second) of a model rocket at a function of time t (in seconds) after launching.

a) When is the rocket speeding up?
b) When does the rocket reach its highest point?
c) When is the rocket's acceleration greatest?

Be sure to give reasons for your answers.



To say the rocket is "speeding up" means that the acceleration (the rate of change of the velocity) is positive. Since the derivative of the velocity will give the slope of the graph of velocity, the acceleration is positive when the slopes are positive. That occurs during the first two and last two seconds, according to the graph.

According to the graph, the rocket is going up (since the velocity is positive) during the first 8 seconds, after which it begins to fall back down; it reaches its highest point at 8 seconds.

As noted above, the acceleration corresponds to the slope of the velocity graph. Judging by the graph, the slope is greatest at 2 seconds.

3. Find an equation for the tangent line to the curve $\tan(xy) = y - (1 + x^2)^{-2}$ at the point $x = 0,\ y = 1$.

Setting $u = xy$ and $v = 1 + x^2$ and differentiating with respect to x:

$$\begin{aligned}
\frac{d\tan u}{dx} &= \frac{dy}{dx} - \frac{dv^{-2}}{dx} \\
\frac{d\tan u}{du}\frac{dxy}{dx} &= y' - \frac{dv^{-2}}{dv}\frac{d1+x^2}{dx} \\
\sec^2 u\left(y\frac{dx}{dx} + x\frac{dy}{dx}\right) &= y' + 2v^{-3}\left(\frac{d1}{dx} + \frac{dx^2}{dx}\right) \\
\sec^2(xy)(y + xy') &= y' + 2(1+x^2)^{-3}(0 + 2x).
\end{aligned}$$

Plugging in $x = 0$ and $y = 1$, we have

$$\begin{aligned}
\sec^2 0(1+0) &= y' + 2(1)^{-3}(0) \\
1 &= y'.
\end{aligned}$$

So the slope of the tangent line is 1. In point-slope form, an equation for the tangent line is $(y - 1) = 1(x - 0)$.

4. A kite 80 feet above the ground moves horizontally as the string is let out at 8 feet per second. How fast is the kite moving when 100 feet of string have been let out? Give a diagram of the situation.

Your diagram should be a right triangle. The height is 80; label the hypotenuse s and the other leg x. Then $\frac{ds}{dt} = 8$ (in feet per second) and we want to find dx/dt at the point when $s = 100$. Using the Pythagorean Theorem, $s^2 = x^2 + 80^2$. Differentiating our equation with respect to t gives

$$\begin{aligned}
\frac{ds^2}{dt} &= \frac{dx^2}{dt} + \frac{d80^2}{dt} \\
\frac{ds^2}{ds}\frac{ds}{dt} &= \frac{dx^2}{dx}\frac{dx}{dt} + 0 \\
2s(8) &= 2x\frac{dx}{dt}.
\end{aligned}$$

Now when $s = 100$ we have $100^2 = x^2 + 80^2$, so $x = 60$. Thus, when $s = 100$ we have $200(8) = 120\frac{dx}{dt}$, $\frac{1600}{120} = \frac{dx}{dt}$; the kite is moving at $1600/120 = 40/3$ feet per second.

5. Compute the following limits.

a. $\lim_{x\to 0} \sec^2 x + \frac{2x}{\sin x}$

Using rules for limits,

$$\begin{aligned}\lim_{x\to 0} \sec^2 x + \frac{2x}{\sin x} &= \lim_{x\to 0} \sec^2 x + 2\lim_{x\to 0} \frac{x}{\sin x} \\ &= \lim_{x\to 0} \sec^2 x + \frac{2}{\lim_{x\to 0} \frac{\sin x}{x}}.\end{aligned}$$

Since the trigonometric functions are continuous on their respective domains, $\lim_{x\to 0} \sec^2 x = \sec^2 0 = 1$. Also, $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Thus,

$$\lim_{x\to 0} \sec^2 x + \frac{2x}{\sin x} = 1 + \frac{2}{1} = 3.$$

b. $\lim_{h\to 0} \frac{\sin^2(\pi/2+h) - 1}{h}$

If $f(x) = \sin^2 x$, then

$$f'(\pi/2) = \lim_{h\to 0} \frac{f(\pi/2+h) - f(\pi/2)}{h} = \lim_{h\to 0} \frac{\sin^2(\pi/2+h) - 1}{h}.$$

On the other hand, using the chain rule with $u = \sin x$ gives

$$f'(x) = \frac{du^2}{du}\frac{d\sin x}{dx} = 2u\cos x = 2\sin x \cos x$$

so $f'(\pi/2) = 2\sin(\pi/2)\cos(\pi/2) = 0$.

It's possible to do this limit using some identities from trigonometry as well:

$$\begin{aligned}\lim_{h\to 0} \frac{\sin^2(\pi/2+h) - 1}{h} &= \lim_{h\to 0} \frac{-\cos^2(\pi/2+h)}{h} \\ &= \lim_{h\to 0} \frac{-(\cos(\pi/2)\cos(h) + \sin(\pi/2)\sin(h))^2}{h} \\ &= \lim_{h\to 0} \frac{-\sin^2 h}{h} \\ &= -\lim_{h\to 0} \frac{\sin h}{h} \lim_{h\to 0} \sin h \\ &= -1\cdot 0 \\ &= 0.\end{aligned}$$

Finally, you should be able to make an "educated guess" for this limit using your calculator.