Practice Problems for Exam #2

Solutions

1. a) Evaluate $\lim_{x\to 0} \frac{\tan x}{4x}$

Use your limit laws and one of the "new" (since last exam) limits we did:

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{4x} &= \frac{1}{4}\lim_{x\to 0}\frac{\tan x}{4} \\ &= \frac{1}{4}\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x} \\ &= \frac{1}{4}\frac{\lim_{x\to 0}\frac{\sin x}{x}}{\lim_{x\to 0}\cos x} \\ &= \frac{1}{4}\frac{1}{1} = \frac{1}{4}.\end{aligned}$$

Here we also use the fact that the cosine function is continuous and $\cos 0 = 1$, so $\lim_{x\to 0}\cos x = \cos 0 = 1$.

b) Find dy/dx and d^2y/dx^2 if $y = x\tan x - \sin^2 x$.

Nothing tricky here; just use our derivative rules and formulas. On the exam **BE SURE TO SHOW ALL OF THE STEPS**!!

$$\begin{aligned}\frac{dy}{dx} &= \frac{dx\tan x}{dx} - \frac{d\sin^2 x}{dx} \\ &= \tan x\frac{dx}{dx} + x\frac{d\tan x}{dx} - \frac{du^2}{du}\,\frac{d\sin x}{dx} \qquad (u = \sin x) \\ &= \tan x + x\sec^2 x - 2u\cos x \\ &= \tan x + x\sec^2 x - 2\sin x\cos x.\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d\tan x}{dx} + \frac{dx\sec^2 x}{dx} - \frac{d2\sin x\cos x}{dx} \\ &= \sec^2 x + \sec^2 x\,\frac{dx}{dx} + x\frac{d\sec^2 x}{dx} - 2\frac{d\sin x\cos x}{dx} \\ &= \sec^2 x + \sec^2 x + x\,\frac{du^2}{du}\frac{d\sec x}{dx} - 2\left(\cos x\frac{d\sin x}{dx} + \sin x\frac{d\cos x}{dx}\right) \qquad (u = \sec x) \\ &= 2\sec^2 x + x(2u)\sec x\tan x - 2(\cos^2 x - \sin^2 x) \\ &= 2\sec^2 x + x2\sec^2 x\tan x - 2\cos^2 x + 2\sin^2 x.\end{aligned}$$

2. A 13 foot ladder leaning up against a wall is sliding down the wall at 1 foot per second.

The ladder makes an angle θ with the ground. How is this angle θ changing when the top of the ladder is five feet from the ground?

Draw an appropriate picture, which in this case will be a right triangle. The base x is the distance from the bottom of the ladder to the wall, the opposite side y is the wall (just from the ground up to the top of the ladder), and the hypotenuse 13 is the ladder. Then $\sin\theta = y/13$. Differentiating with respect to time t gives

$$\begin{aligned}\frac{d\sin\theta}{dt} &= \frac{dy/13}{dt}\\ \frac{d\sin\theta}{d\theta}\frac{d\theta}{dt} &= \frac{1}{13}\frac{dy}{dt}\\ \cos\theta\frac{d\theta}{dt} &= \frac{1}{13},\end{aligned}$$

since we're told that dy/dt is 1 foot per second.

Now when $y = 5$, then by the Pythagorean Theorem $x = 12$ (using $x^2 + y^2 = 13^2$). Since $\cos\theta = x/13$, we get $\cos\theta = 12/13$ when $y = 5$. Plugging that in above, we see that $d\theta/dt = 1/12$ when $y = 5$.

3. Find an equation for the tangent line to the curve $y^2 = 2x^2 - xy$ at the point $(1, 1)$

As always, the derivative gives the slope of the tangent line. To find dy/dx we use implicit differentiation:

$$\begin{aligned}\frac{dy^2}{dx} &= \frac{d2x^2 - xy}{dx}\\ \frac{dy^2}{dy}\frac{dy}{dx} &= \frac{d2x^2}{dx} - \frac{dxy}{dx}\\ 2yy' &= 2\frac{dx^2}{dx} - \left(y\frac{dx}{dx} + x\frac{dy}{dx}\right)\\ 2yy' &= 4x - (y + xy').\end{aligned}$$

At the point $x = 1,\ y = 1$ we have $2y' = 4 - (1 + y')$, so $y' = 1$. That's the slope of the tangent line, so in point-slope form an equation for the tangent line is

$$(y - 1) = 1(x - 1).$$

4. Use differentials to approximate $\sin(\pi/2 - .01)$.

"Use differentials" is just another way of saying "use the tangent line to approximate the graph." So exactly what graph are we approximating here? We are approximating the graph of $y = \sin x$ when x is near $\pi/2$. So first we find the linearization of $\sin x$ at $x = \pi/2$. The derivative is $d\sin x/dx = \cos x$, which is $\cos(\pi/2) = 0$ at $x = \pi/2$. The function value is $\sin(\pi/2) = 1$ at $x = \pi/2$. Thus, an equation for the tangent line at $x = \pi/2$ is

$$(y - 1) = 0(x - \pi/2),$$

so the linearization (just solve for y) is

$$L(x) = 1.$$

Using differentials just means that $\sin(\pi/2 - .01) \approx L(\pi/2 - .01) = 1$.

5. Graph $y = 2\sec^2(2x)$ and $y = \frac{\tan(2x+2h)-\tan(2x)}{h}$ for $h = .1,\ .01$ and $-.001$ together on the same screen. (Use $-\pi/6 \leq x \leq \pi/6$.) What do you see? Can you explain what's going on?

The graphs of $\frac{\tan(2x+2h)-\tan(2x)}{h}$ get closer to the graph of $2\sec^2(2x)$ as h gets closer to 0, which seems to imply that

$$\lim_{h\to 0} \frac{\tan(2x + 2h) - \tan(2x)}{h} = 2\sec^2(2x).$$

Is this really true? Can we really evaluate such a complicated limit? Yes, we can; that limit is a *derivative.* In fact, that limit is precisely the derivative of $\tan(2x)$, and

$$\begin{aligned}\frac{d\tan(2x)}{dx} &= \frac{d\tan u}{du}\frac{d2x}{dx} \qquad (u = 2x)\\ &= \sec^2 u\ 2\frac{dx}{dx}\\ &= 2\sec^2(2x).\end{aligned}$$

The moral of the story here is that all of our rules and formulas for taking derivatives are just "shortcuts" for evaluating certain limits, and those limits can look pretty nasty. Good thing we have those "shortcuts," right?