

Solutions for Exam #3

1. Suppose the first derivative of  $y = f(x)$  is  $y' = x^{1/5}(x+2)$ . Find all critical points, local extrema and points of inflection. Determine where  $f(x)$  is increasing, decreasing, concave up and concave down.

Since  $y' = 0$  when  $x = 0$  and  $x = -2$ , these are the two critical points. You can check that  $y' > 0$  when  $x < -2$  and when  $x > 0$ ; there the function is increasing. The function is decreasing when  $-2 < x < 0$ .

To find  $y''$ , first rewrite  $y' = x^{6/5} + 2x^{1/5}$ . Then

$$y'' = (6/5)x^{1/5} + (2/5)x^{-4/5} = \frac{2}{5}x^{-4/5}(6x + 1),$$

so  $y'' = 0$  when  $x = -1/6$  and  $y''$  doesn't exist when  $x = 0$ . Using the factored form, you see that  $y'' > 0$  when  $-1/6 < x < 0$  and when  $x > 0$ ; there the function is concave up. You can also see that  $y'' < 0$  when  $x < -1/6$ ; there the function is concave down. There is one inflection point at  $x = -1/6$ .

2. Sketch the graph of a function  $f(x)$  which satisfies all the following conditions:

$$f'(x) < 0 \quad x > 1,$$

$$f'(x) > 0 \quad x < 1,$$

$$f''(x) < 0 \quad -1 < x < 2,$$

$$f''(x) > 0 \quad x < -1, \quad x > 2,$$

$$\lim_{x \rightarrow \infty} f(x) = 1.$$

Your graph should be increasing when  $x < 1$  and decreasing when  $x > 1$ . It should be concave up when  $x < -1$  and when  $x > 2$ , and concave down when  $-1 < x < 2$ . Thus, your graph should have a nice local maximum at  $x = 1$  with a horizontal tangent ( $f'$  is continuous on  $(-1, 2)$  since it's differentiable there, so you are forced to have  $f'(1) = 0$ ). Finally, your graph should have a horizontal tangent  $y = 1$  as  $x \rightarrow \infty$ .

3. A farmer has 100 feet of fencing to make a rectangular enclosure for his geese. Assuming a barn will be one side of the enclosure, what is the largest such enclosure he can make and what are its dimensions?

It's best to draw a diagram. Here the diagram is a rectangle. If you label the sides  $x$  and  $y$  with the barn side of length  $y$ , then the perimeter is  $2x + y = 100$ , so  $y = 100 - 2x$ . The area is  $xy = x(100 - 2x) = 100x - 2x^2 = A(x)$ . To maximize  $A(x)$  you set  $A'(x) = 100 - 4x = 0$ . The sole critical point is  $x = 25$ , which does give a maximum by either the first derivative test or the second derivative test (or you could simply look at the graph of  $A(x)$  and see that it is a parabola opening down with vertex at  $x = 25$ ).

The maximum area occurs when  $x = 25$  and  $y = 50$ , for a total area of  $25 \times 50$ .

4. Estimate the area under the graph of  $y = \frac{\sin(\pi x)}{x}$  on  $[1/2, 1]$  by using four rectangles and left endpoints. Include a sketch of the graph with the four rectangles.

See sections 5.1 and 5.2 for similar pictures. For us, the rectangles all have width  $1/8$ . The left endpoints are  $1/2$ ,  $5/8$ ,  $3/4$  and  $7/8$ , so the heights of the four rectangles are

$$\frac{\sin(\pi/2)}{1/2}, \quad \frac{\sin(5\pi/8)}{5/8}, \quad \frac{\sin(3\pi/4)}{3/4}, \quad \frac{\sin(7\pi/8)}{7/8}.$$

The estimate for the area is the sum of the areas of these four rectangles:

$$\frac{1}{8} \left( 2 \sin(\pi/2) + \frac{8 \sin(5\pi/8)}{5} + \frac{4 \sin(3\pi/4)}{3} + \frac{8 \sin(7\pi/8)}{7} \right).$$

5a,b. Find the general antiderivative of the following functions. Show all steps in your work.

a.  $f(x) = \sec x(2 \tan x - \sec x)$

Rewrite this as  $f(x) = 2 \sec x \tan x - \sec^2 x$ . Since the derivative of  $\sec x$  is  $\sec x \tan x$ , an antiderivative of  $\sec x \tan x$  is  $\sec x$ . Since the derivative of  $\tan x$  is  $\sec^2 x$ , and antiderivative of  $\sec^2 x$  is  $\tan x$ . Using the difference and constant multiple rules, the general antiderivative of  $f(x)$  is

$$2 \sec x - \tan x + C.$$

b.  $f(x) = \frac{1 - x^3}{x^2}$

Rewrite as  $f(x) = x^{-2} - x$ . In general, an antiderivative for  $x^n$  is  $x^{n+1}/(n+1)$  (as long as  $n \neq -1$ ). Using this and the difference rule, the general antiderivative of  $f(x)$  is

$$\frac{x^{-1}}{-1} - \frac{x^2}{2}.$$

5c. Evaluate the following definite integral, explaining how you arrive at your answer.

$$\int_0^1 2 + \sqrt{1 - x^2} dx$$

If you draw a graph, you see that this integral is the area of a rectangle surmounted by a quarter circle. The radius of the quarter circle is 1, so its area is  $\pi/4$ . The height of the rectangle is 2 and its width is 1, so its area is 2. The total area is  $2 + \pi/4$ , and that is the value of the integral.