Practice Problems for Final Exam, Part I

Solutions

1. Find the average rate of change of $f(x) = x^2 - 1$ on $[-1, 2]$. Find the instantaneous rate of change at $x = 2$ directly, using limits.

The averate rate of change of a function $f(x)$ on an interval $[a, b]$ is generically $\dfrac{f(b) - f(a)}{b - a}$. In our case here, the average rate of change is

$$\frac{2^2 - 1 - \big((-1)^2 - 1\big)}{2 - (-1)} = 1.$$

The instantaneous rate of chance of a function $f(x)$ at a value a is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

In our case, using the second limit we have

$$\begin{aligned} \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \to 2} \frac{x^2 - 1 - 3}{x - 2} \\ &= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} \\ &= \lim_{x \to 2} x + 2 \\ &= 4. \end{aligned}$$

2. Sketch the graph of a function $y = f(x)$ with all of the following properties:

$$f(0) = 1, \ \lim_{x \to 0^-} f(x) = 2, \ \lim_{x \to \infty} f(x) = 1, \ f'(1) = -1, \ f'(-1) = 0,$$

$f(x)$ is continuous everywhere except $x = 0$, and f is differentiable everywhere except at 0 and 2.

You're on your own here, since posting graphs is a royal pain.

3. Find the following limits. Show all steps.

a) $\displaystyle\lim_{x \to 1} \frac{\sqrt{2x^2 - 1}}{x + 1}$

b) $\displaystyle\lim_{x \to 0} \frac{\tan(x)}{3x}$

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{2x^2-1}}{x+1} &= \frac{\lim_{x\to 1}\sqrt{2x^2-1}}{\lim_{x\to 1}x+1}\\ &= \frac{\sqrt{2(1)^2-1}}{1+1}\\ &= 1/2\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(x)}{3x} &= \lim_{x\to 0}\frac{\frac{\sin(x)}{cos(x)}}{3x}\\ &= \lim_{x\to 0}\frac{\sin(x)}{3x}\lim_{x\to 0}\frac{1}{\cos x}\\ &= \frac{1}{3}\lim_{x\to 0}\frac{\sin(x)}{x}\cdot\frac{1}{\cos 0}\\ &= \frac{1}{3}\cdot\frac{1}{1}.\end{aligned}$$

4. Explain why $f(x) = -x^5 + 2x^2 + x - 1$ has a root. Use Newton's Method to approximate a root to 3 significant digits.

You can check that $f(0) = -1$ and $f(1) = 1$. Since f is continuous, $f(x) = 0$ for some x between 0 and 1 by the Intermediate Value Theorem.

We need $f'(x)$ for Newton's Method: $f'(x) = -5x^4 + 4x + 1$. Starting with an initial "guess" of $x_1 = 1/2$, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1/2 - \frac{f(1/2)}{f'(1/2)} = .5116...$$

and

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = .5115...$$

I'm convinced; there is a root at approximately .512 (rounded to three digits).

5. Compute the derivatives of the following functions, showing all steps.

a) $f(x) = \big(\sin(x) + 3x^2\big)\big(\sqrt{x-1} - 1\big)$

b) $g(x) = \frac{\sec(x)+2x}{\cot(x)+\cos(x)}$

c) $h(x) = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}$

To avoid a big mess, I'm going to compute some derivatives separately. Using the sum rule, constant multiple rule, and a couple formulas,

$$\begin{aligned}\frac{d\sin x+3x^2}{dx} &= \frac{d\sin x}{dx}+\frac{d3x^2}{dx}\\ &= \cos x+3\frac{dx^2}{dx}\\ &= \cos x+6x.\end{aligned}$$

Using the difference rule and the chain rule with $u=x-1$,

$$\begin{aligned}\frac{d\sqrt{x-1}-1}{dx} &= \frac{d\sqrt{x-1}}{dx}-\frac{d1}{dx}\\ &= \frac{du^{1/2}}{du}\cdot\frac{dx-1}{dx}-0\\ &= \frac{1}{2}u^{-1/2}\left(\frac{dx}{dx}-\frac{d1}{dx}\right)\\ &= \frac{1}{2\sqrt{x-1}}(1-0)\\ &= \frac{1}{2\sqrt{x-1}}.\end{aligned}$$

Now using those two derivatives and the product rule,

$$\begin{aligned}\frac{df}{dx} &= \frac{d\sin x+3x^2}{dx}\big(\sqrt{x-1}-1\big)+\big(\sin x+3x^2\big)\frac{d\sqrt{x-1}-1}{dx}\\ &= \big(\cos x+6x\big)\big(\sqrt{x-1}-1\big)+\big(\sin x+3x^2\big)\frac{1}{2\sqrt{x-1}}.\end{aligned}$$

Using the sum and constant multiple rule,

$$\frac{d\sec x+2x}{dx}=\frac{d\sec x}{dx}+\frac{d2x}{dx}=\sec x\tan x+2\frac{dx}{dx}=\sec x\tan x+2.$$

Using just the sum rule,

$$\frac{d\cot x+\cos x}{dx}=\frac{d\cot x}{dx}+\frac{d\cos x}{dx}=-\csc^2 x-\sin x.$$

Using these two derivatives and the quotient rule,

$$\begin{aligned}\frac{dg}{dx} &= \frac{\frac{d\sec x+2x}{dx}\big(\cot x+\cos x\big)-\big(\sec x+2x\big)\frac{d\cot x+\cos x}{dx}}{\big(\cot x+\cos x\big)^2}\\ &= \frac{\big(\sec x\tan x+2\big)\big(\cot x+\cos x\big)+\big(\sec x+2x\big)\big(\csc^2 x+\sin x\big)}{\big(\cot x+\cos x\big)^2}.\end{aligned}$$

Finally, using the chain rule with $u = \sin x$ and the Fundamental Theorem of Calculus,

$$\begin{aligned}
\frac{dh}{dx} &= \frac{d \int_0^u \frac{dt}{\sqrt{1-t^2}}}{du} \cdot \frac{d \sin x}{dx} \\
&= \frac{1}{\sqrt{1-u^2}} \cos x \\
&= \frac{\cos x}{\sqrt{1-\sin^2 x}} \\
&= 1.
\end{aligned}$$

6. Find an equation for the tangent line to the graph of $xy + x^2y^3 = 3x + 3$ at the point $(-1, 1)$.

A line is determined by a point and a slope. We have the point and need to find the slope. The slope is given by the derivative dy/dx, which we will compute using implicit differentiation. Using the same old rules (sum/difference, constant multiple, and product),

$$\begin{aligned}
\frac{dxy + x^2y^3}{dx} &= \frac{d3x + 3}{dx} \\
\frac{dxy}{dx} + \frac{dx^2y^3}{dx} &= \frac{d3x}{dx} + \frac{d3}{dx} \\
\frac{dx}{dx}y + x\frac{dy}{dx} + \frac{dx^2}{dx}y^3 + x^2\frac{dy^3}{dx} &= 3\frac{dx}{dx} + 0 \\
y + xy' + 2xy^3 + x^2\frac{dy^3}{dy}\frac{dy}{dx} &= 3 \\
y + xy' + 2xy^3 + x^2(3y^2)y' &= 3.
\end{aligned}$$

Plugging in $x = -1$ and $y = 1$ gives

$$1 - y' - 2 + 3y' = 3$$

$$2y' = 4$$

$$y' = 2.$$

So the slope is 2. An equation for the tangent line in point-slope form is

$$(y - 1) = 2(x + 1).$$

7. Find the linearization of $f(x) = \sqrt[3]{x}$ at $x = 8$ and use it to approximate $\sqrt[3]{7.9}$.

The linearization is just an equation for the tangent line in $y = mx + b$ form (i.e., y is alone on one side). In particular, the graph of the linearization is the tangent line. So we need the tangent line at $x = 8$. The slope is given by the derivative: $f'(x) = \frac{1}{3}x^{-2/3}$. In our case, the slope is $f'(8) = \frac{1}{12}$. The point is $\big(8, f(x)\big) = (8, 2)$, so the linearization is

$$L(x) = \frac{1}{12}(x - 8) + 2.$$

The linearization approximates the function:

$$f(7.9) \approx L(7.9) = \frac{-.1}{12} + 2 = 2 - \frac{1}{120}.$$