Axioms for Integers

The set of integers, $\mathbb{Z}$, is a non-empty set with a binary operation $+$ satisfying:

1. $(a+b)+c = a+(b+c)$ for any $a, b, c \in \mathbb{Z}$ ($+$ is associative);

2. there is an integer 0 where $0+a = a+0 = a$ for all $a \in \mathbb{Z}$ (there is an additive identity);

3. for every $a \in \mathbb{Z}$ there is a $b \in \mathbb{Z}$ where $a+b = b+a = 0$ (every element has an additive inverse); and

4. $a+b = b+a$ for all $a, b \in \mathbb{Z}$ (additition is commutative).

Also, there is another binary operation $\times$ on $\mathbb{Z}$ which

1. is associative;

2. has an identity not equal to the additive identity;

3. distributes through addition on both the left and right, i.e., $a \times (b+c) = a \times b + a \times c$ and $(a+b) \times c = a \times c + b \times c$ for all $a, b, c \in \mathbb{Z}$; and

4. is commutative.

Further, there is a order relation $\leq$ on $\mathbb{Z}$ which totally orders $\mathbb{Z}$, i.e.,

1. $a \leq a$ for all $a \in \mathbb{Z}$;

2. if $a \leq b$ and $b \leq a$, then $a = b$;

3. if $a \leq b$ and $b \leq c$, the $a \leq c$; and

4. for all $a, b \in \mathbb{Z}$, either $a \leq b$ or $b \leq a$.

This order relation satisfies:

1. if $a \leq b$, then $a+c \leq b+c$ for all $c \in \mathbb{Z}$;

2. if $a \leq b$ and $0 \leq c$, then $a \times c \leq b \times c$; and

3. $0 < 1$. (Here $<$ means $\leq$ and not equal.)

Finally, every non-empty subset of $\mathbb{N} = \{a \in \mathbb{Z} \colon 0 \leq a\}$ has a least element.