

Ideals and Another Look

at Theorem 1.1.4

Definition: A non-empty subset I of the integers is called an *ideal* if it has the following three properties:

1. if $a \in I$, then $-a \in I$;
2. if $a, b \in I$, then $a + b \in I$; and
3. if $a \in I$, then $az \in I$ for all integers z .

Lemma: If I is an ideal, then $0 \in I$.

Theorem 1.1.4 redux: If I is an ideal, then I consists of all integer multiples of some number.

In other words, $I = a\mathbb{Z} = \{a \cdot z : z \in \mathbb{Z}\}$.

IMPORTANT: You can prove this using the “Division Algorithm” (Theorem 1.1.3) and the fact that an ideal with more than one element has a non-zero element of least absolute value.