The Degree Function for Polynomials

Suppose $P(X) = a_n X^n + \cdots + a_0 \in \mathbb{Q}[X]$ and $a_n \neq 0$. Then the *degree* of $P(X)$ is n. The degree of 0 is defined to be $-\infty$. We write $\deg(P(X))$ (or just $\deg(P)$) for the degree of the polynomial $P(X)$.

Aside: for our purposes, we'll say $-\infty < n$ and $-\infty + n = -\infty$ for all integers n.

Lemma: For any two polynomails $P(X)$ and $Q(X)$,

$$\deg(P \times Q) = \deg(P) + \deg(Q)$$

and

$$\deg(P + Q) \leq \max\{\deg(P), \deg(Q)\}.$$

This is the degree/polynomial analog of the following result for integers.

Lemma: For any two integers a and b,

$$|a \cdot b| = |a| \cdot |b|$$

and

$$|a + b| \leq |a| + |b|.$$

We can make things look even more the same by defining an "absolute value" on polynomials by $|P(X)| = e^{\deg(P)}$ (with the convention that $e^{-\infty} = 0$).

Anything we can prove using the axioms for integers for addition and multiplication, together with the properties of the absolute value and the degree above, will be valid simultaneously for *both* the integers and polynomials (and anything else with the same properties, for that matter).

Lemma: For polynomials $P(X)$ and $Q(X)$, if $P(X) \times Q(X) = 0$, then either $P(X) = 0$ or $Q(X) = 0$. For integers a and b, if $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Proof: First for the integer case, suppose neither a nor b are 0. Then $|a|$ and $|b|$ are not zero, hence at least 1 (by our last "little result" for integers). In particular, we have

$$0 < 1 \leq |a| = |a| \cdot 1| \leq |a| \cdot |b|.$$

Since $|a| \cdot |b| = |a \cdot b| > 0$, we see that $a \cdot b \neq 0$.

Now for the polynomial case. (NOTE: we need the integer case, in fact more, for the polynomial case.) Suppose neither $P(X)$ nor $Q(X)$ are 0. Then $\deg(P)$ and $\deg(Q)$ are both at least 0 (the only polynomial of degree less than 0 is 0). Thus $\deg(P \times Q) = \deg(P) + \deg(Q) \geq 0 + 0 = 0$ and $P(X) \times Q(X) \neq 0$. You can make this look even more like the proof for integers by either exponentiating the degree or taking logarithms of the absolute value.

The following can now be proven for polynomials exactly as we can for integers (these were on the list of "little results").

Lemma: If $P(X) \times Q(X) = P(X) \times R(X)$ and $P(X) \neq 0$, then $Q(X) = R(X)$.

Lemma: The multiplicative identity is unique.