Some Solutions for Week 10 Homework

17. Let G be a group and suppose $H_i \subseteq G$ is a subgroup of G for all $i \in I$. Let $H = \bigcap_{i\in I} H_i$. Since the identity element $e \in H_i$ for all i, $e \in H$ and H is not empty. Now suppose $a, b \in H$. Then $a, b \in H_i$ for all i and so $ab^{-1} \in H_i$ for all i since each H_i is a subgroup. Thus $ab^{-1} \in H$ and H is a subgroup by Corollary 3.2.3.

21. b) This is essentially just restating the definition set-theoretically:

$$Z(G) = \{x \in G\colon xa = ax \text{ for all } a \in G\} = \bigcap_{a\in G} \{x \in G\colon xa = ax\} = \bigcap_{a\in G} C(a).$$

a) Follows from b), #17 and #19 (which we did in class).

24. a) Let $a \in G$. Then by #19 from section 3.1, $(a^{-1})^n = a^{-n} = (a^n)^{-1}$ for all integers n. Since $e^{-1} = e$, we see that $(a^{-1})^n = e$ if and only if $a^n = e$. This shows that the order of a^{-1} is equal to the order of a (even if it is infinite).

b) Let $a, b \in G$ and suppose m is a positive integer. Then by associativity

$$(ab)^m = \underbrace{(ab)(ab)\cdots(ab)}_{m \text{ times}} = a\,\underbrace{(ba)(ba)\cdots(ba)}_{m-1 \text{ times}}\,b = a(ba)^{m-1}b.$$

Now suppose $(ab)^n = e$. Then by what we just showed,

$$aeb = ab = abe = (ab)(ab)^n = (ab)^{n+1} = a(ba)^n b,$$

so that $(ba)^n = e$ by right and left cancellation. This shows that the order of ba is no greater than the order of ab. Of course, this argument is entirely symmetric (just switch the roles of a and b), so that the order of ab is no greater than the order of ba. Thus, they have the same order (even if it is infinite).

c) Using b) and associativity, for any $a, b \in G$ we have

$$o(aba^{-1}) = o\big(a(ba^{-1})\big) = o\big((ba^{-1})a\big) = o\big(b(a^{-1}a)\big) = o(b).$$

8. Suppose G_1 and G_2 are groups with identity elements e_1 and e_2, and subgroups H_1 and H_2, respectively. Then $e_1 \in H_1$ and $e_2 \in H_2$, so that $(e_1, e_2) \in H_1 \times H_2$. In particular $H_1 \times H_2$ is

not empty. Let (g_1, g_2) and (h_1, h_2) be elements of $H_1 \times H_2$. Then $g_1, h_1 \in H_1$ and $g_2, h_2 \in H_2$, so that $g_1 h_1^{-1} \in H_1$ and $g_2 h_2^{-1} \in H_2$. Thus,

$$(g_1, g_2) \cdot (h_1, h_2)^{-1} = (g_1, g_2) \cdot (h_1^{-1}, h_2^{-1}) = (g_1 h_1^{-1}, g_2 h_2^{-1}) \in H_1 \times H_2$$

and $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$ by Corollary 3.2.3.