Isomorphisms II

Two groups are isomorphic if and only if everything group-wise about them is the same. Suppose $G_1 \cong G_2$. Then

- They have the same order.
- Their elements have the same order.
- They have the same (isomorphic) subgroups.
- They have the same center.

It's just as likely that you'll want to show two groups are *not* isomorphic. If so, the above points are good to keep in mind.

Examples: 1) If G is a group of order 6, then either $G \cong \mathbb{Z}_6$ or $G \cong S_3$, but not both.

2) If G is a group of order 4, then either $G \cong \mathbb{Z}_4$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, but not both.

3) If G is a group of order p where p is a prime number, then $G \cong \mathbb{Z}_p$.

4) The groups $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, D_4 and Q (the quaternions) are all groups of order 8, but no two of these groups are isomorphic.

Now suppose two groups G_1 and G_2 are isomorphic. For example,

$\mathbb{Z}_6$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$. There is an isomorphism $\phi\colon \mathbb{Z}_6 \to \mathbb{Z}_2 \times \mathbb{Z}_3$:

$$\phi(n[1]_6) = n([1]_2, [1]_3) \qquad \text{for } n \in \mathbb{Z}.$$

Is there another?

Suppose you have two isomorphisms $\sigma,\ \tau\colon G_1 \to G_2$. Then τ^{-1} is an isomorphism from G_2 to G_1 and $\tau^{-1} \circ \sigma$ is an isomorphism from G_1 to itself.

If $\phi\colon G_1 \to G_1$ is an isomorphism, then $\sigma \circ \phi$ is another isomorphism from G_1 to G_2.

Definition: If G is a group, the set of isomorphisms from G to itself is a group with composition as the binary operation. This is called the *automorphism group* of G and is typically denoted $A(G)$. Note that this is actually a subgroup of $\mathrm{Sym}(G)$, the one-to-one and onto functions of the *set* G to itself.

What are the automorphism groups for the following groups?

- $\mathbb{Z}$
- $\mathbb{Z}_6$
- $\mathbb{Z}_5$
- $\mathbb{Z}_n$
- S_3
- $\mathbb{Z}_2 \times \mathbb{Z}_2$