Direct Products

Definition: Suppose G_1 and G_2 are groups. Then the set of ordered pairs (a, b) where $a \in G_1$ and $b \in G_2$ form a group, called the *direct product* of G_1 and G_2, with binary operation given by

$$(a, b) \cdot (c, d) = (ac, bd).$$

Here ac is the "product" of a and c in the group G_1 and bd is the "product" of b and d in the group G_2. In particular, please note that these "products" may not be alike at all.

The direct product of G_1 and G_2 is denoted $G_1 \times G_2$.

Examples: 1) The direct product $\mathbb{Z}_2 \times \mathbb{Z}_2$ is an abelian group with four elements called the *Klein four group.* It is abelian, but not cyclic.

2) More generally, the direct product $\mathbb{Z}_m \times \mathbb{Z}_n$ is an abelian group with mn elements. Sometimes it is cyclic.

3) The direct product $S_3 \times S_4$ is a group with $6 \times 24 = 144$ elements. It isn't abelian.

4) The direct product $S_3 \times \mathbb{Z}_n$ is a group with $6n$ elements. It isn't abelian.

5) The direct product $S_3 \times \mathbb{Z}$ is a group with infinitely many elements. It isn't abelian.

Lemma: The direct product $G_1 \times G_2$ of two groups is abelian if and only if both G_1 and G_2 are abelian.

Proof: Suppose $G_1 \times G_2$ is abelian. Let $a, b \in G_1$ and let $e_2 \in G_2$ be the identity element of G_2. Then

$$(ab, e_2) = (a, e_2) \cdot (b, e_2) = (b, e_2) \cdot (a, e_2) = (ba, e_2),$$

so that $ab = ba$. Let $c, d \in G_2$ and let $e_1 \in G_1$ be the identity element of G_1. Then

$$(e_1, cd) = (e_1, c) \cdot (e_1, d) = (e_1, d) \cdot (e_1, d) = (e_1, dc),$$

so that $cd = dc$. Thus, G_1 and G_2 are both abelian.

Now suppose both G_1 and G_2 are abelian and let $(a, b), (c, d) \in G_1 \times G_2$. Then

$$(a, b) \cdot (c, d) = (ac, bd) = (ca, db) = (c, d) \cdot (a, b),$$

so $G_1 \times G_2$ is abelian.

What is the identity element of $G_1 \times G_2$?

What is the inverse of an element $(a, b) \in G_1 \times G_2$?

What are the subgroups of $G_1 \times G_2$?

What prevents us from going nuts here and getting a direct product of three groups:

$$G_1 \times G_2 \times G_3?$$

Or four groups? Or even infinitely many groups?

Or even elements arranged in matrix form?!?