SOME GROUPS OF ORDER 4

Suppose G is a group of order 4. Then by Lagrange's theorem, an element of G has order 1, 2 or 4. Of course, the only element with order 1 is the identity. If there is an element of order 4, then G is cyclic and essentially the integers modulo 4 (its multiplication table will look exactly like the addition table for the integers modulo 4).

So let's suppose for that no element of G has order 4. Then all elements have order 1 or 2. In other words, $x^2 = e$ for all $x \in G$. By a previous exercise, G is abelian. Have we seen such a group? Indeed we have: the set of 4 matrices from exercise # 8 from section 1.1,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Label these matrices $e,\ a,\ b$ and c. Then the multiplication table for this group is as follows:

$\times$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

We will get the *exact same multiplication table* if we look at the subgroup of S_4 consisting of the permutations $(1),\ (1,2)(3,4),\ (1,3)(2,4)$ and $(1,4)(2,3)$ and label these $e,\ a,\ b$ and c.

Here is yet another way to get the same multiplication table. Consider ordered pairs of elements of the integers modulo 2. There are four such ordered pairs (where, as usual, I've left off the brackets):

$$(0,0), \qquad (0,1), \qquad (1,0), \qquad (1,1)$$

We make this collection into a group by adding them together in the most obvious manner: $(a,b)+(c,d)=(a+c,b+d)$. Let's check that we get the same table as above. (Here it would be an "addition table," I suppose.)