

Dihedral Groups

We've seen the Dihedral groups D_n as “rigid motions” of the regular n -gon, whence as subgroups of the symmetric groups S_n . For a fixed $n > 2$, we saw that D_n has order $2n$, with n rotations and n “flips.”

First, we can show that D_n is never abelian. To see why, consider the rotation by $2\pi/n$, which is represented by the n -cycle $\sigma = (1, 2, \dots, n)$, and the “flip” you get when holding the first corner, which is represented by a product of disjoint transpositions $\tau = (2, n) \cdots$. If we “flip” first then rotate, the first corner ends up at 2. You can also see this by considering what the composition $\sigma \circ \tau$ (remember, these are functions on the set $\{1, 2, \dots, n\}$) does to 1: τ leaves 1 fixed and σ sends 1 to 2. However, if we rotate first then “flip”, the first corner ends up at n : $\tau \circ \sigma(1) = \tau(2) = n$. This implies that the rotation σ and the “flip” τ do not commute.

Next we consider the multiplication table for D_n in much the same way we did for S_3 . Let g denote the rotation σ above (to keep the notation similar to what we did for S_3 before) and let f denote the “flip” τ . Then we can write the elements of D_n as

$$e, g, g^2, \dots, g^{n-1}, f, fg, \dots, fg^{n-1}.$$

Note that these elements are distinct. The powers of g are the n rotations, and the rest are “flips.” As with the multiplication table for S_3 , we will be able to fill in the multiplication table for D_n once we know what gf is.

Now gf is not a power of g , since f isn't. Also $gf \neq fg$ since otherwise we would see that D_n is abelian (which it isn't). In fact, we saw above that the “flip” followed by the rotation sends corner 1 to 2. We know this must be a “flip” of some kind, since it isn't a rotation. There is exactly one “flip” in D_n that sends corner 1 to 2. Since fg^{n-1} (this is rotating *backwards* by $2\pi/n$ followed by our “flip” τ) sends corner 1 to 2, this must be the same “flip” as gf . In other words, $gf = fg^{n-1}$.

We are thus lead to a more “algebraic” (if you will) definition of the dihedral group D_n . It is the non-abelian group of order $2n$ gotten by taking an element g of order n , an element f of order 2 which is not equal to any power of g , and setting $gf = fg^{n-1} = fg^{-1}$.