Groups of Order 6

Suppose G is a group of order 6. What can G "look like?"

We know from Lagrange's theorem that elements of G may only have order 1, 2, 3 or 6. Only the identity element has order 1 (by the definition of order).

Suppose G has an element of order 6. Then G is cyclic, and its multiplication table is essentially the same as the addition table for $\mathbb{Z}_6$.

Suppose G doesn't have any elements of order 6. Since the identity has order 1, this leaves five other elements of order 2 or 3. By a previous exercise, the number of elements of order 3 is even. Since 5 isn't even, there must be at least one element of order 2.

Suppose all elements of G besides the identity have order 2. Then by a previous exercise G is abelian. Take two distinct elements of order 2; call them a and b. Since G is abelian,

$$H = \{e,\ a,\ b,\ ab\}$$

is a subgroup of G. But this can't be right, since Lagrange's theorem says the order of any subgroup of G must divide 6, and H has order 4.

Summarizing what we have so far, if G isn't cyclic, then it has an element of order 2 and an element of order 3; call them f and g. Note that $f = f^{-1}$ and $g^2 = g^{-1}$.

Now if $fg = gf$ the powers of fg are:

$$(fg)^1 \qquad (fg)^2 = f^2g^2 = g^2 \qquad (fg)^3 = f^3g^3 = f$$

$$(fg)^4 = f^4g^4 = g \qquad (fg)^5 = f^5g^5 = fg^2 \qquad (fg)^6 = f^6g^6 = e$$

Thus, G is cyclic after all! This means that if G isn't cyclic, then it has an element f of order 2 and an element g of order 3, and $fg \neq gf$. In particular, G must not be abelian.

If G isn't cyclic, then we now know what the elements of G are:

$$G = \{e,\ f,\ g,\ g^2,\ fg,\ gf\}.$$

What can the multiplication table for G look like?

Conclusion: If G is a group of order 6, then either G is cyclic and essentially $\mathbb{Z}_6$ or G isn't abelian at all and is essentially S_3. Here "essentially" means that, after relabelling the elements of G, we get the exact same "multiplication" table.