

Some Groups which are not the Same

In contrast to what we just looked at, consider the following three groups:

$$\mathbb{Z}_8, \quad \mathbb{Z}_4 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

All three of these groups have eight elements. Are these groups really “the same?” Are any two of them “the same?” In other words, would the addition tables be the same after renaming the elements?

The answer here is no. The reason isn’t too hard to see: the group \mathbb{Z}_8 has an element of order 8 (four of them, in fact), and neither of the other two have an element of order 8; the group $\mathbb{Z}_4 \times \mathbb{Z}_2$ has four elements of order 4, but $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has only elements of order 2 and an identity element.

Thus, these three groups have three clearly different addition tables.

Now consider the group D_4 from the midterm. It is the following subgroup of S_4 :

$$D_4 = \{(1), (1, 2, 3, 4), (1, 3)(1, 4), (1, 4, 3, 2), (1, 2)(3, 4), (2, 4), (1, 4)(2, 3), (1, 3)\}.$$

Even though this group has eight elements, it can’t be the same as either of the first three since it isn’t abelian:

$$(1, 2)(3, 4)(1, 2, 3, 4) = (2, 4) \quad (1, 2, 3, 4)(1, 2)(3, 4) = (1, 3).$$

Finally, the quaternion group Q is another non-abelian group of order 8:

$$Q = \{\pm 1, \pm i, \pm j, \pm k\}.$$

This group is definitely *not* the same as D_4 since this group has only one element of order 2, namely -1 .

Are there any other groups of order 8?

If G has an element of order 8, then G must be cyclic and the same as \mathbb{Z}_8 .

If every element $x \in G$ satisfies $x^2 = e$, then G must be abelian (by a previous exercise). In this case, G must be the same as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

So suppose G has no element of order 8, but at least one element of order 4. Call that element g . We then get a cyclic subgroup of G with four elements:

$$H = \langle g \rangle = \{e = g^0, g^1, g^2, g^3\}.$$

Recalling how we proved Lagrange's theorem, we can partition the group G into two left cosets: H and fH , where f is an element of G not in H . In other words,

$$G = \{e = g^0, g, g^2, g^3, f, fg, fg^2, fg^3\}$$

We will be able to completely fill in the multiplication table once we know f^2 and gf .

By left cancellation, we see that f^2 is *not* equal to f, fg, fg^2 or fg^3 since $f \notin H$. In other words, f^2 must be e, g, g^2 or g^3 . On the other hand, the order of f isn't 8, so f^2 must be either e or g^2 . In the first case, f has order 2. In the second case f has order 4.

Next, we see that gf must be in fH since $f \notin H$. By right cancellation, $gf \neq f$. If $gf = fg^2$, then multiplying by g^2 on the right gives $gfg^2 = f$ (since g has order 4), and multiplying this by g^3 on the left gives $fg^2 = g^3f$. This is a contradiction, since we now have $gf = g^3f$.

In summary, either $f^2 = e$ or $f^2 = g^2$, and either $gf = fg$ or $gf = fg^3$.

Suppose $fg = gf$ and $f^2 = e$. Then we can fill in the multiplication table, and it looks just like the addition table for $\mathbb{Z}_4 \times \mathbb{Z}_2$. (Write g for $([1]_4, [0]_2)$ and f for $([0]_4, [1]_2)$, for example.)

If $fg = gf$ and $f^2 = g^2$, then the order of fg is 2 since $(fg)^2 = f^2g^2 = g^4 = e$. The multiplication table for G can now be completed just like we did above; let $h = fg$ and replace each f in the first table with h . (Notice that $hg = gh$ and $h^2 = e$.)

Suppose $gf = fg^3$ and $f^2 = e$. Then we can completely fill in the multiplication table, and it's just like the composition table for D_4 . (Write g for $(1, 2, 3, 4)$ and f for $(1, 2)(3, 4)$, for example.)

If $gf = fg^3$ and $f^2 = g^2$, then we can completely fill in the multiplication table, and it's just like the composition table for Q . (Write g for i and f for j , for example.)

In conclusion, there are really only 5 different groups of order 8. Three of them are abelian and two are non-abelian. They are completely determined by the number of elements of order 2 and whether or not they are abelian.