

## More on Cycles and Permutations

As noted before, a *transposition* is a 2-cycle.

Notice how any  $m$ -cycle can be written as a product (composition) of  $m - 1$  transpositions:

$$(i_1, i_2, \dots, i_m) = (i_1, i_m)(i_1, i_{m-1}) \cdots (i_1, i_2).$$

Now this way of writing a cycle as a product of transpositions is just one way of doing so.

There are many different ways.

**Examples:**

$$(2, 5, 3, 6) = (2, 6)(2, 3)(2, 5) = (5, 2)(3, 5)(6, 3) = (1, 7)(2, 6)(2, 3)(2, 5)(1, 7).$$

$$(1) = (1, 3)(1, 3) = (1, 2)(1, 2)(3, 5)(3, 5).$$

What we want to show is that, while a cycle (or any type of permutation) may be written in different ways as a product of transpositions, and even using different numbers of transpositions, the *parity* of the number of transpositions used stays the same. In other words, if a permutation can be written as the product of an odd number of transpositions, then it can only be written as the product of an odd number of transpositions. If a permutation can be written as the product of an even number of transpositions, then it can only be written as a product of an even number of transpositions.

How could we do this? How should we think about doing something like this?

What we need is a way to think of permutations as either “even” or “odd.” In other words, we need a suitable **function** which takes permutations as input and has either 1 or  $-1$ , say, as output.

Start with  $n$  variables:  $X_1, X_2, \dots, X_n$ . Take the product

$$\prod_{i < j} (X_i - X_j)$$

and call this product  $P$ . When  $n = 4$ , this product is

$$P = (X_1 - X_2)(X_1 - X_3)(X_1 - X_4)(X_2 - X_3)(X_2 - X_4)(X_3 - X_4).$$

Generally speaking, there will be  $n(n-1)/2$  factors in this product. For every pair of variables  $X_i$  and  $X_j$ , either  $X_i - X_j$  or  $X_j - X_i$  is a factor of  $P$ , but not both.

Since the subscripts on the variables are just the numbers 1 through  $n$ , we can view a permutation as a rearrangement of the variables. In other words, we can view a permutation  $\sigma$  as changing the product  $P$ :

$$P_\sigma = \prod_{i < j} (X_{\sigma(i)} - X_{\sigma(j)}).$$

**Example:** Suppose  $n = 4$  again and  $\sigma$  is the cycle  $(1, 3, 4, 2)$ . Then

$$P_\sigma = (X_3 - X_1)(X_3 - X_4)(X_3 - X_2)(X_1 - X_4)(X_1 - X_2)(X_4 - X_2).$$

Just like  $P$ , the product  $P_\sigma$  will have  $n(n-1)/2$  factors. For every pair of variables  $X_i$  and  $X_j$ , either  $X_i - X_j$  or  $X_j - X_i$  will be a factor of  $P_\sigma$ . In other words, **either  $P_\sigma$  is  $P$ , or it is  $-P$ .**

Define the function  $f: S_n \rightarrow \{1, -1\}$  by the rule

$$P_\sigma = f(\sigma)P.$$

Certainly the identity function won't change  $P$ , so  $f$  of the identity function is 1. What about the next simplest type of permutation, transpositions?

**Example:** Take  $n = 4$  again and suppose  $\sigma = (1, 3)$ . Then

$$P_\sigma = (X_3 - X_2)(X_3 - X_1)(X_3 - X_4)(X_2 - X_1)(X_2 - X_4)(X_1 - X_4) = -P.$$

$$f(\sigma) = -1.$$

Generally speaking, a transposition  $(a, b)$  will only affect those factors involving  $X_a$  or  $X_b$ . Of course, there is one factor involving both of them:  $(X_a - X_b)$  (assuming that  $a < b$ ). How many other factors will  $(a, b)$  affect? There are  $n - 2$  factors involving  $X_a$  but not  $X_b$ , and  $n - 2$  factors involving  $X_b$  but not  $X_a$ .

Suppose (as we may) that  $a < b$ . Of those factors involving  $X_a$  but not  $X_b$ ,  $(a, b)$  will only change the sign of the factor when the index on the other variable is between  $a$  and  $b$ . But the same can be said of those factors involving  $X_b$  but not  $X_a$ . In other words, the combined effect of the transposition  $(a, b)$  on all the factors involving just one of  $X_a$  or  $X_b$  is nothing, since there are an even number of sign changes.

Of course it's easy to see the effect of  $(a, b)$  on the factor  $(X_a - X_b)$ : it changes it to  $(X_b - X_a)$ .

**Theorem:** If  $\sigma$  is a transposition, then  $f(\sigma) = -1$ .

And now for the really crucial step.

**Lemma:** For any two elements  $\sigma, \tau \in S_n$ ,

$$f(\sigma \circ \tau) = f(\sigma)f(\tau).$$

(not really a) **Proof:** You can, if you like, simply check the four possible cases for  $f(\sigma)$  and  $f(\tau)$ . For example, suppose  $f(\sigma) = 1$  and  $f(\tau) = -1$ . Then applying  $\tau$  to  $P$  first changes it to  $-P$ . Applying  $\sigma$  to  $-P$  won't change it, since  $\sigma$  doesn't change  $P$ . So the combined effect of the composition  $\sigma \circ \tau$  is to change the sign of  $P$ .

**Corollary:** A permutation  $\sigma \in S_n$  can be written as a product of an even number of transpositions if and only if  $f(\sigma) = 1$ . It can be written as a product of an odd number of transpositions if and only if  $f(\sigma) = -1$ .

**Remark:** As mentioned before, an  $m$ -cycle can be written as a product of  $m-1$  transpositions. Therefore, an  $m$ -cycle is even if and only if  $m$  is odd.

What are the even permutations in  $S_3$ ? What are the even permutations in  $S_4$ ?

Are exactly half of the permutations in  $S_n$  even?