

Sets and Axioms

Typically we just assume we know what a “set” is and what “element of a set” means. That seems harmless enough, but does leave some rather nasty loose ends. (Look up “Russel’s Paradox” on the internet to see one such loose end.)

There is a more satisfactory answer. We can, in a manner similar to our dealing with the integers before, not define sets per se, but instead list the basic properties which sets should enjoy. These basic properties, or axioms, are usually taken to be the Zermelo-Fraenkel axioms together with the axiom of choice: ZFC for short. Here they are together with brief editorials.

ZF1: Two sets are equal if and only if they have the same elements.

ZF2: There is a set with no elements.

[This is called the empty set or null set, and written \emptyset . Be sure you don’t confuse this with the letter phi, ϕ .]

ZF3: Given any two sets x and y , there is a set whose elements are x and y .

ZF4: Given any set x whose elements are sets, there is a set which has as its elements all elements of elements of x .

[This allows us to take the union of sets. For example, suppose x and y are sets. Then ZF3 says that $\{x, y\}$ is a set, so that there is a set $x \cup y$ via ZF4.]

ZF5: Given any set x , there is a set whose elements are the subsets of x .

[This is called the power set axiom, since the set whose existence it asserts is called the power set of x . For example, suppose $x = \{1, 2\}$. Then the power set of x , typically written $P(x)$, is $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. It’s an interesting exercise to show that $P(x)$ contains 2^n elements if x has n elements.]

ZF6: Given any well-formed formula $\mathcal{A}(y)$ in ZF expressing an assertion about the set y , and given any set x , there is a set whose elements are all the elements $y \in x$ for which $\mathcal{A}(y)$ is true. In other words, there is a set $\{y \in x : \mathcal{A}(y)\}$.

ZF7: Let $\mathcal{F}(x, y)$ be a well-formed formula of ZF which determines a function. Then, given any set x , there is a set v consisting of all images of elements of x under the function \mathcal{F} .

[These two axioms are crucial for “resolving” things like Russell’s paradox. Of course, the

devil is in the details here; what is a “well-formed formula” in ZF? This can be made precise. The idea is to only allow the building of “formulas” using things of the form $a \in b$, $a = b$, conjunctions (“and” statements), disjunctions (“or” statements), negation, implication and quantifiers (“for all” and “there exists”). By itself, ZF6 doesn’t allow the construction of sets whose elements are listed by a particular “rule.” This is why ZF7 is necessary. It turns out that ZF7 implies ZF6, meaning the listing of ZF6 is redundant. It’s usually listed simply because it’s the way one usually goes about constructing sets.]

ZF8: There is a set x such that $\emptyset \in x$, and such that for every set $u \in x$, we also have $u \cup \{u\} \in x$.

[This is called the axiom of infinity since it asserts the existence of (an) infinite set(s).]

ZF9: Every non-empty set x contains an element which is disjoint from x .

[This one isn’t really used in mathematics so much, but it’s nice to “know” that, for example, no set can be an element of itself.]

AC: Given any non-empty set x whose elements are pairwise disjoint non-empty sets, there is a set which contains precisely one element from each element of x .

[This is the notorious axiom of choice. I say notorious since, while seemingly innocuous, it (and more specifically its consequences) definitely bothers some. Much effort has been spent seeing how much “math” one can develop without it. The vast majority of mathematicians use it unquestioningly. Feel free to do the same.]

So there you have it. Those are the properties which “sets” are supposed to enjoy. If your interest is piqued by this sort of thing, I can suggest a few books to read.