Infinite Series Summary

1. Special series to remember:

- **Geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1}$$

Here a is the first term and r is the common ratio.
When $|r| < 1$, the series is convergent, and we have:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

When $|r| \geq 1$, the series is divergent.

- **The harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is a famous divergent series with $a_n = 1/n \longrightarrow 0$. This example reminds us that $a_n \longrightarrow 0$ alone cannot guarantee the convergence of the corresponding series $\sum_{n=1}^{\infty} a_n$.

- **The p series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

When $p > 1$, the series is convergent.
When $p \leq 1$, the series is divergent.
Note that when $p = 1$ the p series has another name: the harmonic series.

- **The alternating series**
$\sum_{n=1}^{\infty} (-1)^n b_n$ with $b_n > 0$
When $b_{n+1} \leq b_n$ and $b_n \longrightarrow 0$, it is convergent.
When $b_n \not\longrightarrow 0$, it is divergent. So the alternating harmonic series:

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

is convergent.

- Note: (1) When the ratio of a geometric series is negative, it is also an alternating series. In this case, it is much more productive to treat it as geometric series.
(2) For alternating series, be aware of the difference between a_n and b_n. It is a very

helpful habit to write down the b_n explicitly.
(3) To verify b_n is decreasing: Suppose that when we substitute x for n in b_n we get a function $f(x)$, where $b_n = f(n)$. You can show $f(x)$ is a decreasing function by determining that its derivative $f'(x)$ is on an interval (a, ∞). (Usually the endpoint a will be 1 or 2.)

2. Convegence Tests for a series $\displaystyle\sum_{n=1}^{\infty} a_n$:

- **nth term test:**

  If $\displaystyle\lim_{n\to\infty} a_n$ does not exist or exists but the limit is not zero, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

  Note that this test cannot be used to show convergence.

- **Integral Test:** Only for $a_n \geq 0$
  Suppose that replacing n by x in a_n results in a function $f(x)$, where $f(n) = a_n$.

  Form the improper integral $\quad I = \displaystyle\int_1^{\infty} f(x)dx$.

  I converges $\Longrightarrow \displaystyle\sum_{n=1}^{\infty} a_n$ converges

  I diverges $\Longrightarrow \displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

- **Simple Comparison Test:** Only for $a_n \geq 0$

  You must use an appropriate positive series $\displaystyle\sum_{n=1}^{\infty} b_n$ for which you know its convergence.

  (a) $a_n \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges $\Longrightarrow \displaystyle\sum_{n=1}^{\infty} a_n$ converges

  (b) $a_n \geq b_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges $\Longrightarrow \displaystyle\sum_{n=1}^{\infty} a_n$ diverges

- **Limit Comparison Test:** Only for $a_n \geq 0$

  Choose an appropriate positive series $\displaystyle\sum_{n=1}^{\infty} b_n$ for which you know its convergence.

  $\displaystyle\lim_{n\longrightarrow\infty} \frac{a_n}{b_n} = L > 0 \Longrightarrow$ series $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ have the same convergency.

- **Ratio Test:** Useful when factorials appear in a_n.

  Compute $\displaystyle\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = L$

  $L < 1 \Longrightarrow$ series $\displaystyle\sum_{n=1}^{\infty} a_n$ *absolutely* converges

  $L > 1 \Longrightarrow$ series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges

- **Root Test:** Useful when a_n contains factors raised to the power n.
  Compute $\lim\limits_{n\longrightarrow\infty} \sqrt[n]{|a_n|} = L$

  $L < 1 \implies$ series $\sum\limits_{n=1}^{\infty} a_n$ *absolutely* converges

  $L > 1 \implies$ series $\sum\limits_{n=1}^{\infty} a_n$ diverges

3. The Race to Infinity. How to guess the convergence of a given series in the form: $\sum\limits_{n=1}^{\infty} a_n$ where $a_n = \dfrac{c_n}{d_n}$ and $\lim\limits_{n\longrightarrow\infty} c_n = \lim\limits_{n\longrightarrow\infty} d_n = \infty$ with c_n and d_n both positive?

   - First we note that the convergence of $\sum\limits_{n=1}^{\infty} a_n$ somewhat depends on the relative speed of how fast c_n and d_n approach infinity.

     For example, n^3+1 approaches ∞ faster than n^2 and n do, but $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^3+1}$ converges and $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{n^3+1}$ diverges. The reason? The relative speeds of approaching ∞ are different: $n^3+1 \to \infty$ is faster relative to $n \to \infty$ as to $n^2 \to \infty$.
   - The following is an incomplete list of $f(n)$ with the speed approaching infinity from the slowest to the fastest:

     $$\ln n \prec n^p \prec r^n \prec n! \prec n^n$$

     where $p > 0$ and $r > 1$ are constants.
     (The expression $f(n) \prec g(n)$ means $\lim\limits_{n\to\infty} \dfrac{f(n)}{g(n)} = 0$.)
     Try to guess the convergency of series:
     $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^p}$, $\sum\limits_{n=1}^{\infty} \dfrac{1}{n \ln^p n}$, $\sum\limits_{n=1}^{\infty} \dfrac{1}{n \ln n (\ln(\ln n))^p}$, $\sum\limits_{n=1}^{\infty} \dfrac{n^p}{r^n}$, $\sum\limits_{n=1}^{\infty} \dfrac{r^n}{n!}$, and $\sum\limits_{n=1}^{\infty} \dfrac{n!}{n^n}$
     Test your guess for each!
   - What if $c_n \to \infty$ faster than $d_n \to \infty$? The original series diverges.

4. Absolute and conditional convergence:

   - Series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent $\iff$ series $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent.
   - Series $\sum\limits_{n=1}^{\infty} a_n$ is conditionally convergent $\iff$

     series $\sum\limits_{n=1}^{\infty} a_n$ is convergent, but series $\sum\limits_{n=1}^{\infty} |a_n|$ is divergent.

- From above definitions we know that the only series that could be conditionally convergent is the series having infinitely many negative terms and infinitely many positive terms. In this book, that is the alternating series. Thus, the only series that could be the candidate for conditionally convergent is the alternating series (in this text book).
- Any series that is tested to be convergent by using the tests listed in 2 must be absolutely convergent. (Why?)

Examples:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

   - $p \neq 1$:
     Use the integral test: Integration by parts ($u = \ln x$, $dv = x^{-p}dx$) gives
     $\displaystyle\int_2^t \ln x x^{-p} dx = \frac{1}{-p+1}\Big((\ln x)x^{-p+1}|_2^t - \int_2^t x^{-p+1}\frac{1}{x}dx\Big)$
     $\displaystyle= \frac{1}{-p+1}\Big((\ln x)x^{-p+1}|_2^t - \int_2^t \frac{1}{x^p}dx\Big).$
     When $t \to \infty$, if $p > 1, x^{-p+1}|_2^t = \dfrac{1}{t^{p-1}} - \dfrac{1}{2^{p-1}} \to -\dfrac{1}{2^{p-1}}$ and the integral $\displaystyle\int_2^t \frac{1}{x^p}dx$ converges too; if $p < 1$, they both diverge. So it is convergent when $p > 1$ and divergent if $p < 1$.
   - $p = 1$:
     Use the Integral test or else compare it with the harmonic series. It will be divergent.
   - Conclusion: The original series convergent when $p > 1$ and divergent when $p \leq 1$.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

   - Use the ratio test:
     $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!n^n}{n!(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \Big(\frac{n+1}{n}\Big)^{-n} = \frac{1}{e}$
     The last equation comes from a famous limit: $\displaystyle\lim_{n\to\infty} \Big(\frac{n+1}{n}\Big)^n = e$, which can be shown by H'lopital's rule.
     Since $1/e < 1$, according to the test the original series converges.

3. Determine whether the series $-\dfrac{2}{3} + \dfrac{2}{9} - \dfrac{2}{27} + \dfrac{2}{81} - \cdots$ converges or diverges. If it converges find its exact sum.

   - We recognize this as an example of both an alternating series and a geometric series. However, since the problem also asks us the exact sum, we should treat it as geometric series with first term $a = -\frac{2}{3}$ and ratio $r = -\frac{1}{3}$.