$\mathrm{M}_\vee$- matrices : A generalization of M-matrices based on eventually nonnegative matrices

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This is joint work with Michael Tsatsomeros and Pauline van den Driessche.

Notation and Definitions

Given $X \in \mathbb{R}^{n \times n}$, the spectrum of X is denoted by $\sigma(X)$ and its spectral radius by $\rho(X) = \max\{|\lambda| \mid \lambda \in \sigma(X)\}$.

An $n \times n$ matrix $B = [b_{ij}]$ is *nonnegative* (*positive*), denoted by $B \geq 0$ ($B > 0$), if $b_{ij} \geq 0$ ($b_{ij} > 0$) for all i and j.

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B is *eventually nonnegative* (*positive*), denoted by $B \overset{\text{v}}{\geq} 0$ ($B \overset{\text{v}}{>} 0$) , if there exists a nonnegative integer k_0 such that $B^k \geq 0$ ($B^k > 0$) for all $k \geq k_0$.

We denote the smallest such nonnegative integer by $k_0 = k_0(B)$ and refer to it as the *power index* of B.

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An $n \times n$ matrix $A = [a_{ij}]$ is called

- an *M-matrix* if $A = sI - B$, where $B \geq 0$ and $s \geq \rho(B) \geq 0$
- an *M_{v}-matrix* if $A = sI - B$, where $B \overset{\mathrm{v}}{\geq} 0$ and $s \geq \rho(B) \geq 0$

Notation and Definitions

Definition

A matrix $X \in \mathbb{R}^{n \times n}$ has

- the *Perron-Frobenius property* if $\rho(X) > 0$, $\rho(X) \in \sigma(X)$ and there exists a nonnegative eigenvector corresponding to $\rho(X)$;
- the *strong Perron-Frobenius property* if, in addition to having the Perron-Frobenius property, $\rho(X)$ is a simple eigenvalue such that

  $$\rho(X) > |\lambda| \quad \text{for all} \quad \lambda \in \sigma(X),\ \lambda \neq \rho(X)$$

  and the corresponding eigenvector is strictly positive.

Known results

A result of Johnson and Tarazaga (2004)

For a matrix $B \in \mathbb{R}^{n \times n}$, the following are equivalent:

(i) Both matrices B and B^T have the strong Perron-Frobenius property.

(ii) B is eventually positive.

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A result of Noutsos (2006) and Elhashash/Szyld (2006)

Let $B \in \mathbb{R}^{n \times n}$ be an eventually nonnegative matrix that is not nilpotent. Then both B and B^T have the Perron-Frobenius property.

Basic properties

Some obvious properties of $\mathrm{M}_{\vee}$- matrices (analogous to M-matrices)

If $A = sI - B$ is an $\mathrm{M}_{\vee}$- matrix, then
(i) $s - \rho(B) \in \sigma(A)$
(ii) $\mathrm{Re}\,\lambda \geq 0$ for all $\lambda \in \sigma(A)$
(iii) $\det A \geq 0$ and $\det A = 0$ if and only if $s = \rho(B)$
(iv) if, in particular, $\rho(B) > 0$, then there exists an eigenvector $x \geq 0$ of A and an eigenvector $y \geq 0$ of A^T corresponding to $\lambda(A) = s - \rho(B)$
(v) if, in particular, $B \overset{\mathrm{v}}{>} 0$ and $s > \rho(B)$, then in (iv) $x > 0$, $y > 0$ and in (ii) $\mathrm{Re}\,\lambda > 0$ for all $\lambda \in \sigma(A)$

Exponential nonnegativity

Definition

An $n \times n$ matrix $B = [b_{ij}]$ is called

- *exponentially nonnegative* (*positive*) if $\forall t \geq 0$,
  $$e^{tB} = \sum_{k=0}^{\infty} \frac{t^k B^k}{k!} \geq 0 \ \ (e^{tB} > 0)$$
- *eventually exponentially nonnegative* (*positive*) if $\exists t_0 \in [0, \infty)$ such that $\forall t \geq t_0$, $e^{tB} \geq 0$ $(e^{tB} > 0)$

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For an M-matrix A, clearly $-A + \alpha I \geq 0$ for sufficiently large $\alpha \geq 0$, implying that

$$e^{-tA} = e^{-t\alpha} e^{-t(A - \alpha I)} \geq 0 \text{ for all } t \geq 0.$$

That is $-A$ is exponentially nonnegative.

Eventual exponential positivity

An extension of the above property to $\mathrm{M}_\vee$- matrices .

Theorem

Let $A = sI - B \in \mathbb{R}^{n\times n}$ *be an* $M_\vee$*- matrix with* $B \overset{\text{v}}{>} 0$ (*and thus* $s \geq \rho(B) > 0$). *Then* $-A$ *is eventually exponentially positive. That is, there exists* $t_0 \geq 0$ *such that* $e^{-tA} > 0$ *for all* $t \geq t_0$.

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Proof.

Let $A = sI - B$, where $B = sI - A \overset{\vee}{>} 0$ with power index k_0. As $B^m > 0$ for all $m \geq k_0$, there exists sufficiently large $t_0 > 0$ so that for all $t \geq t_0$, the sum of the first $k_0 - 1$ terms of the series $e^{tB} = \displaystyle\sum_{m=0}^{\infty} \frac{t^m B^m}{m!}$ is dominated by the term $\dfrac{t^{k_0} B^{k_0}}{k_0!}$, rendering e^{tB} positive for all $t \geq t_0$. It follows that $e^{-tA} = e^{-ts} e^{tB}$ is positive for all $t \geq t_0$. That is, $-A$ is eventually exponentially positive as claimed. ☐

Index and eventual exponential nonnegativity

To obtain a similar result with eventual exponential nonnegativity, we need a definition:

The degree of 0 as a root of the minimal polynomial of A is denoted by $\mathrm{index}_0(A)$, and if A is nonsingular, then $\mathrm{index}_0(A) = 0$.

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A result of Noutsos and Tsatsomeros (2008)

Let $B \in \mathbb{R}^{n \times n}$ such that $B \overset{\mathrm{v}}{\geq} 0$ and $\mathrm{index}_0(B) \leq 1$. Then B is eventually exponentially nonnegative.

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Theorem

Let $A = sI - B \in \mathbb{R}^{n\times n}$ be an $M_\vee$- matrix where $B \overset{\mathrm{v}}{\geq} 0$ and $\mathrm{index}_0(B) \leq 1$. Then $-A$ is an eventually exponentially nonnegative matrix.

Proof.

The result follows readily from the above result and the fact that $e^{-tA} = e^{-ts}e^{tB}$. ☐

Index and eventual exponential nonnegativity

Corollary

Let $A = sI - B \in \mathbb{R}^{n\times n}$ *be an* $M_\vee$*- matrix such that* $B + \alpha I \overset{v}{\geq} 0$ *for some* $\alpha \in \mathbb{R}$ *with* $-\alpha \notin \sigma(B)$. *Then* $-A$ *is an eventually exponentially nonnegative matrix.*

Proof.

Since $A = (s+\alpha)I - (B + \alpha I)$ is an $M_\vee$- matrix and $B + \alpha I \overset{v}{\geq} 0$, it follows that $s + \alpha \geq \rho(B + \alpha I)$. Since $\mathrm{index}_0(B + \alpha I) = 0 \leq 1$, the corollary follows by applying the latter theorem to $A = (s+\alpha)I - (B+\alpha I)$. ☐

An example

Consider $A = 3I - B$, where

$$B = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Since B is eventually nonnegative with $\rho(B) = 2$, A is an $\mathrm{M}_\vee$-matrix. As $\mathrm{index}_0(B) = 1$, it follows that $-A$ is eventually exponentially nonnegative.

A second example

When $\mathrm{index}_0(B) > 1$, the conclusion of the last theorem is not in general true.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

is eventually nonnegative with $k_0(B) = 2$ and $\mathrm{index}_0(B) = 2$. Let $A = sI - B$ with $s \geq \rho(B)$. As

$$B^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} & k2^{k-1} & k2^{k-1} \\ 2^{k-1} & 2^{k-1} & k2^{k-1} & k2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \end{bmatrix} \quad (k = 2, 3, \ldots),$$

it follows that the $(3,1)$ and $(4,2)$ entries of e^{tB} (and thus e^{-tA}) are negative for all $t > 0$. That is, $-A$ is not eventually exponentially nonnegative.

Monotonicity, semipositivity and inverse nonnegativity

The following well known properties of M-matrices:

monotonicity, semipositivity and inverse nonnegativity

are now examined in the context of $\mathrm{M}_{\vee}$- matrices.

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Theorem

Let $A = sI - B \in \mathbb{R}^{n\times n}$, *where* $B \overset{v}{\geq} 0$ *has power index* $k_0 \geq 0$. *Let the cone* K *be defined as* $K = B^{k_0}\mathbb{R}^n_+$. *Consider the following conditions:*

(i) *A is an invertible $M_\vee$- matrix*

(ii) $s > \rho(B)$ (positive stability of A)

(iii) A^{-1} *exists and* $A^{-1}K \subseteq \mathbb{R}^n_+$ (inverse nonnegativity)

(iv) $Ax \in K \implies x \geq 0$ (monotonicity)

Then (i)$\Longleftrightarrow$(ii)$\Longrightarrow$(iii)$\Longleftrightarrow$(iv).

If, in addition, B is not nilpotent, then all conditions (i)-(iv) *are equivalent.*

Monotonicity, semipositivity and inverse nonnegativity

Remarks

(a) The above implication (iii)$\Longrightarrow$(i) is not in general true if B is nilpotent. For example, consider $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \overset{\text{v}}{\geq} 0$, which has power index $k_0 = 2$. Thus $K = B^2\mathbb{R}^2_+ = \{0\}$. For any $s < 0$, $A = sI - B$ is invertible and $A^{-1}K = \{0\} \subset \mathbb{R}^2_+$. However, A is not an $\mathrm{M}_\vee$- matrix as its eigenvalues are negative.

Monotonicity, semipositivity and inverse nonnegativity

(b) It is well known that when an M-matrix is invertible, its inverse is nonnegative. Johnson and Tarazaga (2004) show that the inverse of a pseudo M-matrix is eventually positive. Le and McDonald (2006) show that if B is an irreducible eventually nonnegative matrix with $index_0(B) \leq 1$, then there exists $t > \rho(B)$ such that for all $s \in (\rho(B), t)$, $(sI - B)^{-1} > 0$.

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The situation with the inverse of an $\mathrm{M}_\vee$- matrix A is different. Notice that condition (iii) of the last theorem is equivalent to $A^{-1}B^{k_0} \geq 0$. In general, if A is an invertible $\mathrm{M}_\vee$- matrix, A^{-1} is neither nonnegative nor eventually nonnegative.

Monotonicity, semipositivity and inverse nonnegativity

For a subclass of the $\mathrm{M}_{\vee}$- matrices , we have the following semipositivity result.

Theorem

Let $A = sI - B \in \mathbb{R}^{n\times n}$, *where* $B \overset{\mathrm{v}}{\geq} 0$ *has a positive eigenvector* (*corresponding to* $\rho(B)$). *Consider the following conditions:*

(i) *A is an $M_{\vee}$- matrix*

(ii) *There exists an invertible diagonal matrix $D \geq 0$ such that the row sums of AD are nonnegative*

(iii) *There exists $x > 0$ such that $Ax \geq 0$* (semipositivity)

Then (i)$\Longrightarrow$(ii)$\Longleftrightarrow$(iii).

If, in addition, B is not nilpotent, then all conditions (i)-(iii) *are equivalent.*

Monotonicity, semipositivity and inverse nonnegativity

Remarks

(a) Let $A = -\frac{1}{4}\, I + B$, with $B = \left[\begin{array}{cc} 1 & -1 \\ 1 & -1 \end{array}\right] \overset{\mathrm{v}}{\geq} 0$, and $x = [2,1]^T$.
Then (iii) above holds, but A is not an $\mathrm{M}_{\vee}$- matrix. Thus the implication (iii)$\Longrightarrow$(i) is not in general true if B is nilpotent.

Monotonicity, semipositivity and inverse nonnegativity

Remarks

(a) Let $A = -\frac{1}{4} I + B$, with $B = \left[\begin{array}{cc} 1 & -1 \\ 1 & -1 \end{array} \right] \overset{\mathrm{v}}{\geq} 0$, and $x = [2, 1]^T$. Then (iii) above holds, but A is not an $\mathrm{M}_\vee$- matrix. Thus the implication (iii)$\Longrightarrow$(i) is not in general true if B is nilpotent.

(b) The existence of a positive eigenvector in the above theorem is necessary. To see this, consider $B = \left[\begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array} \right] \overset{\mathrm{v}}{\geq} 0$, which is nilpotent and has no positive eigenvector. Let

$$A = \frac{1}{4} I - B = \left[\begin{array}{cc} -\frac{3}{4} & -1 \\ 1 & \frac{5}{4} \end{array} \right].$$

Notice that there is no $x > 0$ such that $Ax \geq 0$, but A is an $\mathrm{M}_\vee$-matrix. That is, (i) of the above theorem holds but not (iii).

Monotonicity, semipositivity and inverse nonnegativity

An invertible M-matrix can be scaled to be diagonally dominant: $ADe > 0$. But (ii) of the above theorem does not imply this for $\mathrm{M}_\vee$- matrices because off-diagonal entries of $B \overset{\mathrm{v}}{\geq} 0$ can be negative. For example, let

$$A = sI - B = 9.5\,I - \begin{bmatrix} -0.1 & 20 & 47 \\ -0.2 & 1 & 1 \\ 0.3 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 9.6 & -20 & -47 \\ 0.2 & 8.5 & -1 \\ -0.3 & -5 & 1.5 \end{bmatrix}.$$

The matrix A is an invertible $\mathrm{M}_\vee$- matrix because $B \overset{\mathrm{v}}{>} 0$ with $\rho(B) = 9.4834$. Letting $D = \mathrm{diag}(w)$, where $w = [0.9799, 0.0004, 0.1996]^T$ is an eigenvector of B corresponding to $\rho(B)$, gives

$$AD = \begin{bmatrix} 9.4068 & -0.0086 & -9.3819 \\ 0.1960 & 0.0036 & -0.1996 \\ -0.2940 & -0.0021 & 0.2994 \end{bmatrix}.$$

Note that $ADe \geq 0$, however, AD is not diagonally dominant.

Monotonicity, semipositivity and inverse nonnegativity

Next we give some properties of singular $\mathrm{M}_{\vee}$- matrices analogous to properties of singular M-matrices.

Theorem

Let $A = sI - B \in \mathbb{R}^{n \times n}$ *be a singular* $M_{\vee}$*- matrix, where* $B \overset{\mathrm{v}}{>} 0$*. Then the following hold.*

(i) A *has rank* $n-1$*.*

(ii) *There exists a vector* $x > 0$ *such that* $Ax = 0$*.*

(iii) *If for some vector* u*,* $Au \geq 0$*, then* $u = 0$ (almost monotonicity)*.*

Monotonicity, semipositivity and inverse nonnegativity

The following is a comparison condition for $\mathrm{M}_\vee$- matrices analogous to a known result for M-matrices (see e.g., Horn and Johnson).

Theorem

Let $A = sI - B \in \mathbb{R}^{n\times n}$ *and* $E = sI - F \in \mathbb{R}^{n\times n}$*, where* $B, F \overset{\mathrm{v}}{\geq} 0$ *are not nilpotent. Suppose that at least one of* B*,* B^T*,* F *or* F^T *has a positive eigenvector (corresponding to the spectral radius). If* A *is an* $M_\vee$*- matrix and* $A \leq E$*, then* E *is an* $M_\vee$*- matrix.*

Monotonicity, semipositivity and inverse nonnegativity

An important aspect of M-matrix theory is principal submatrix inheritance: every principal submatrix of an M-matrix is also an M-matrix.

As a consequence, all principal minors of an M-matrix are nonnegative (i.e., every M-matrix is a P_0-matrix).

These facts do not carry over to $M_\vee$- matrices as seen in the next example.

Monotonicity, semipositivity and inverse nonnegativity

Consider

$$B = \begin{bmatrix} 9.5 & 1 & 1.5 \\ -14.5 & 16 & 10.5 \\ 10.5 & -3 & 4.5 \end{bmatrix}$$

for which $\rho(B) = 12$ is a simple dominant eigenvalue having positive left and right eigenvectors. That is, B and B^T satisfy the strong Perron-Frobenius property and so $B \overset{\text{v}}{>} 0$. As a consequence,

$$A = 12.5\,I - B = \begin{bmatrix} 3 & -1 & -1.5 \\ 14.5 & -3.5 & -10.5 \\ -10.5 & 3 & 8 \end{bmatrix}$$

is an invertible $\mathrm{M}_\vee$- matrix. Clearly, A is not a P_0-matrix since the $(2,2)$ entry is negative. Also the $(2,2)$ entry is a principal submatrix of A that is not an $\mathrm{M}_\vee$- matrix.