Connectivity Constrained Reference Basis Model Updating

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LAYOUT

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Summary

Second Order Systems Basics

A linear mechanical system with n degrees of freedom (dof) is modeled by

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t) \quad M > 0, \quad C, K \geq 0$$

The eigenvalue equation

$$\left(M\lambda^2 + C\lambda + K\right)\phi = 0$$

Assumption – no damping, i.e. C=0. The pencil becomes first order and the problem is generalized eigenvalue problem

$$\left(K - M\bar{\lambda}\right)\phi = 0 \quad , \quad \bar{\lambda} = -\lambda^2 = \omega^2$$

For PD M and K the eigenvalues of the linear pencil are positive and real, therefore those of the original problem are purely imaginary $\pm j\omega$.

Second Order Systems Basics (cont.)

Define

$$\Phi_k = [\phi_1 \quad \cdots \quad \phi_k] \quad , \quad \Omega = diag\{\omega_1 \quad \cdots \quad \omega_k\}$$

Then

$$K\Phi_k = M\Phi_k \Omega_k{}^2$$

The modeshapes are orthogonal w.r.t both M and K.

With appropriate normalization

$$\Phi^T M \Phi = I \quad , \quad \Phi^T K \Phi = \Omega^2$$

An immediate result is

$$K = M\Phi_n \Omega_n{}^2 \Phi_n{}^T M = M\Phi_1 \Omega_1{}^2 \Phi_1{}^T M + M\Phi_2 \Omega_2{}^2 \Phi_2{}^T M$$

The Model Updating Problem

The 'true' model (n dof's)

$$M_T\ddot{x}(t)+K_Tx(t)=f(t)$$

The analytic model (typically FE)

$$M_A\ddot{x}(t)+K_Ax(t)=f(t)$$

The measurements – *m* natural frequencies ω_i and (possibly partial) modeshapes ϕ_i.

$$\Omega_E(m\times m) \quad , \quad \Phi_E(l\times m)$$

Problem (general): combine the analytic information M_A, K_A and the experimental results Ω_E, Φ_E to obtain a model M, K which is more accurate, i.e. closer to M_T, K_T.

Mathematical Problem

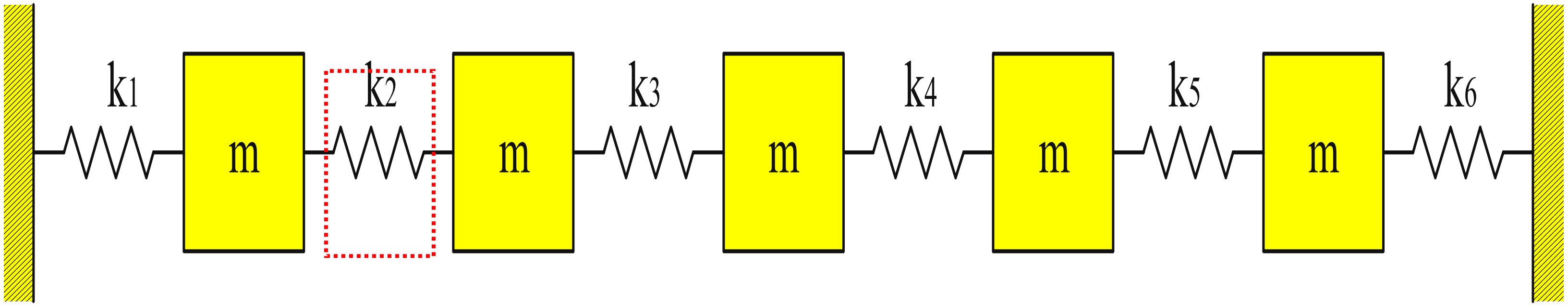
Parameterization:

$$K(\alpha) = K(\alpha_1, \cdots, \alpha_p)$$

or in linear form

$$K(\alpha) = K_A + \sum_{i=1}^{p} \alpha_i K_i$$

Connectivity and Placeholder Matrices



The stiffness matrix has a clear connectivity.
If α_i represents the deviation of k_2 from its nominal value, K_i is given by

$$K_A = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 & 0 \\ 0 & 0 & -k_4 & k_4+k_5 & -k_5 \\ 0 & 0 & 0 & -k_5 & k_5+k_6 \end{bmatrix}$$

$$K_i = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Mathematical Problem

Parameterization:

$$K(\alpha) = K(\alpha_1, \cdots, \alpha_p)$$

or in linear form

$$K(\alpha) = K_A + \sum_{i=1}^{p} \alpha_i K_i$$

Problem: Given M>0, $\Phi(n\times m)$ and diagonal $\Omega(m\times m)$ find parameters α_1, …α_p such that

$$K(\alpha)\Phi = M\Phi\Omega^2$$

Solution: Exact solution requires $p \geq m(n-(m-1)/2)$. If p is smaller, a measure of not satisfying the equation is minimize.

Minimizing the Error in the Characteristic Equation (MECE)

Type: Equation error.

The optimization criterion

$$\min_{\alpha} \; J_{MECE} = \sum_{i=1}^{p} (K(\alpha)\phi_i - M\phi_i\omega_i^2)^T W_i (K(\alpha)\phi_i - M\phi_i\omega_i^2)$$

For identical W_i it reduces to

$$\min_{\alpha} \; J_{MECE} = trace\left\{(K(\alpha)\Phi - M\Omega^2)^T W (K(\alpha)\Phi - M\Omega^2)\right\}$$

This is a standard LS problem with a solution via linear equations.

Sensitivity Methods

(Natke, Lallement, Link, Mottershead, Friswell …)

Type: Error in the solution.

The optimization criterion

$$\min_{\alpha} \; (z_E - z_A(\alpha))^T W (z_E - z_A(\alpha)) + \alpha^T R \alpha$$

The iterative solution is based on repeated linearizations

$$\alpha^{(k+1)} = \alpha^{(k)} + \left(S_k^T W S_k + R\right)^{-1} S_k^T W \left(z_E - z_A(\alpha^{(k)})\right) \qquad \alpha^{(0)} = 0$$

$$S_k = \frac{\partial z}{\partial \alpha} \Big|_{\alpha = \alpha^{(k)}}$$

S_k is called the sensitivity matrix:

Field Error

(Ladevèze et al)

Type: Error in the displacement field.

The optimization criterion (simplified)

$$\min_{\alpha,v,u} J = \sum_{i=1}^{m} \|v_i - u_i\|_2^2 + w_i \|v_i - \phi_i\|_2^2$$

$$s.t. \quad K(\alpha)v_i = M u_i \omega_i^2$$

Ideally

$$v_i = u_i = \phi_i$$

Parametric Methods

Advantages

- Connectivity is preserved.
- Physically meaningful parameters.
- User can decide which parameters to use.

Disadvantages

- Large amount of calculation.
- Eigendata is not matched exactly.
- User should decide which parameters to use.

The Reference Basis Method

Baruch and Bar Izthack (1978), Berman and Nagy (1983), Kammer (1983), Beatie and Weaver-Smith (1992).

- Non-parametric.
- Experimental data – A partial set of natural frequencies (accurate) and modeshapes (inaccurate)
- Some quantities (typically the mass) are assumed to be accurate.
- A minimal deviation of the others from the analytical model is sought, so that
- The eigenvalues equation is satisfied exactly.

Standard Reference Basis

(Baruch and Bar Izthack,1978)

Step 1- Orthogonalization such that $\Phi^T M \Phi = I_m$

Step 2- Stiffness matrix updating

$$\min_K J = \left\| M^{-1/2} \left(K - K_A \right) M^{-1/2} \right\|_F^2$$

$$s.t. \quad K = K^T$$

$$K\Phi = M\Phi\Omega^2$$

Solution

$$K = K_A - K_A \Phi\Phi^T M - M\Phi\Phi^T K_A + M\Phi\Phi^T K_A \Phi\Phi^T M + M\Phi\Omega^2\Phi^T M$$

Standard Reference Basis

Advantages

- The resulting model in compatible with the eigendata.
- Closed form solution
- Small amount of calculation.

Disadvantages

- No relationship with physical parameters of the model.
- Connectivity is not preserved.
- No mechanism for iteration or user’s intervention.

Connectivity Constrained Reference Basis

The goal of the connectivity constrained reference basis method is to combine the two approaches.

In particular, modifying the Reference Basis method by introducing physical notions into it, but still maintaining its theoretical and numerical advantages

Generalized Reference Basis

(Kenigsbuch and Halevi, 1998)

$$\min_{L} \quad J = \left\| \mathrm{W}^{-1/2} \left(K - \mathrm{K_A} \right) \mathrm{W}^{-1/2} \right\|_F^2$$

$$s.t \qquad K = \mathrm{L}\mathrm{L}^T$$

$$K\Phi = \mathrm{M}\Phi\Omega^2$$

W - an arbitrary positive definite matrix

Solution: "Kalman Filter" form

$$K = K_A - \left(K_A \Phi - M\Phi\Omega^2 \right) R^T - R\left(K_A \Phi - M\Phi\Omega^2 \right)^T + R\left(\Phi^T K_A \Phi - \Omega^2 \right) R^T$$

$$R = W_k \Phi \left(\Phi^T W_k \Phi \right)^{-1}$$

Generalized Reference Basis (cont.)

Solution: "geometric" form

$$K = M\Phi\Omega^2\Phi^T M + P\left(K_A - M\Phi\Omega^2\Phi^T M\right)P^T$$

$$P = I - R\Phi^T = I - W\Phi\left(\Phi^T W\Phi\right)^{-1}\Phi^T$$

P is a projection into the subspace

$$U = null\left(\Phi^T\right) = span\left(M\tilde{\Phi}\right)$$

Recalling that

$$K_T = M\Phi\Omega^2\Phi^T M + M\tilde{\Phi}\tilde{\Omega}^2\tilde{\Phi}^T M$$

P projects the second term into the required subspace.

W determines only the "angle' of the projection

Weighting Matrix Selection

$$J = \left\| W^{-1/2}(K - K_A)W^{-1/2} \right\|_F^2$$

Observation : W_{ii} 'large' $\Rightarrow$ change in the *i*-th dof is 'cheap'.

let d_i be proportional to the error in the i-th dof, found from any error localization method. For example

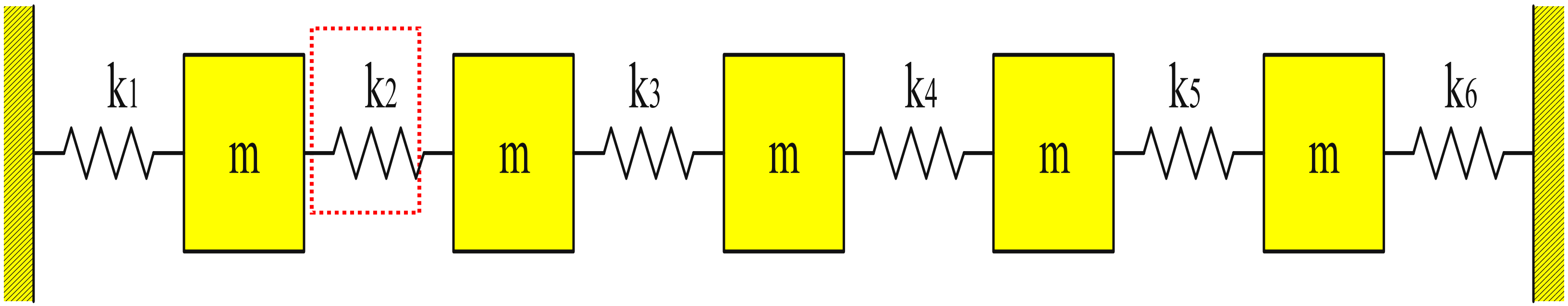
$$d_i = c_i \left\| \left(M\Phi\Omega^2 - K_A\Phi \right)_{i,*} \right\|_2$$

Then use

$$W = \mathrm{diag}\{d_i^{\beta}\}$$

Where β is a design parameter.

Connectivity – Motivating Example



Analytic model : $m=1,\ k_1= k_2=50 = k_3= k_4= k_5=k_6=50.$

True model : $m=1,\ k_1= k_3= k_4= k_5=k_6=50,\ k_2=70.$

$$\Omega_T=\{\ 3.7532,\ 7.0711,\ 10.2090,\ 12.8027,\ 14.4146\ \}$$

$$\Omega_A=\{\ 3.6603,\ 7.0711,\ 10.0000,\ 12.2474,\ 13.6603\ \}$$

Example (cont.)

β=0

K=

97.8790	-50.6112	**-1.9625**	**-1.7271**	**-1.0051**
-50.6112	103.0060	-48.2799	**1.5138**	**0.8810**
-1.9625	-48.2799	99.8259	-50.1532	**-0.0892**
-1.7271	**1.5138**	-50.1532	99.8652	-50.0785
-1.0051	**0.8810**	**-0.0892**	-50.0785	99.9543

β=0.3

K=

120.0000	-70.0000	0.0000	0.0000	0.0000
-70.0000	120. 0000	-50. 0000	0.0000	0.0000
0.0000	-50. 0000	100. 0000	-50. 0000	0.0000
0.0000	0.0000	-50. 0000	100.0000	-50. 0000
0.0000	0.0000	0.0000	-50. 0000	100. 0000

A posteriori Connectivity Assignment

Step 1: solve Generalized RB problem for the unconstrained K.

Step 2: apply connectivity to K.

$$\begin{aligned}&\min \quad J_1 = \left\|K_{con} - K\right\|_F^2 \\ &s.t. \\ &K_{con} = \mathrm{K_A} + \sum_{i=1}^{p} \alpha_i \mathrm{K_i}\end{aligned} \quad \Rightarrow \quad \min_{\alpha} \quad J_1 = \left\| \sum_{i=1}^{p} \alpha_i \mathrm{K_i} - (K - \mathrm{K_A}) \right\|_F^2$$

Standard least squares, has a closed form for the normal equation

$$\begin{bmatrix} tr\left(K_1 K_1\right) & \cdots & tr\left(K_1 K_p\right) \\ \vdots & \ddots & \vdots \\ tr\left(K_p K_1\right) & \cdots & tr\left(K_p K_p\right) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} tr\left(K_1 \left(K - K_A\right)\right) \\ \vdots \\ tr\left(K_p \left(K - K_A\right)\right) \end{bmatrix}$$

The Cost of Connectivity

Unlike K, K_{con}, does not satisfy the eigenvalue equation.

$$\Omega_{con} = \Omega\,(M, K_{con}) \neq \Omega \qquad , \qquad \Phi_{con} = \Phi\,(M, K_{con}) \neq \Phi$$

The deviation of K_{con}, from K is the **cost of connectivity**

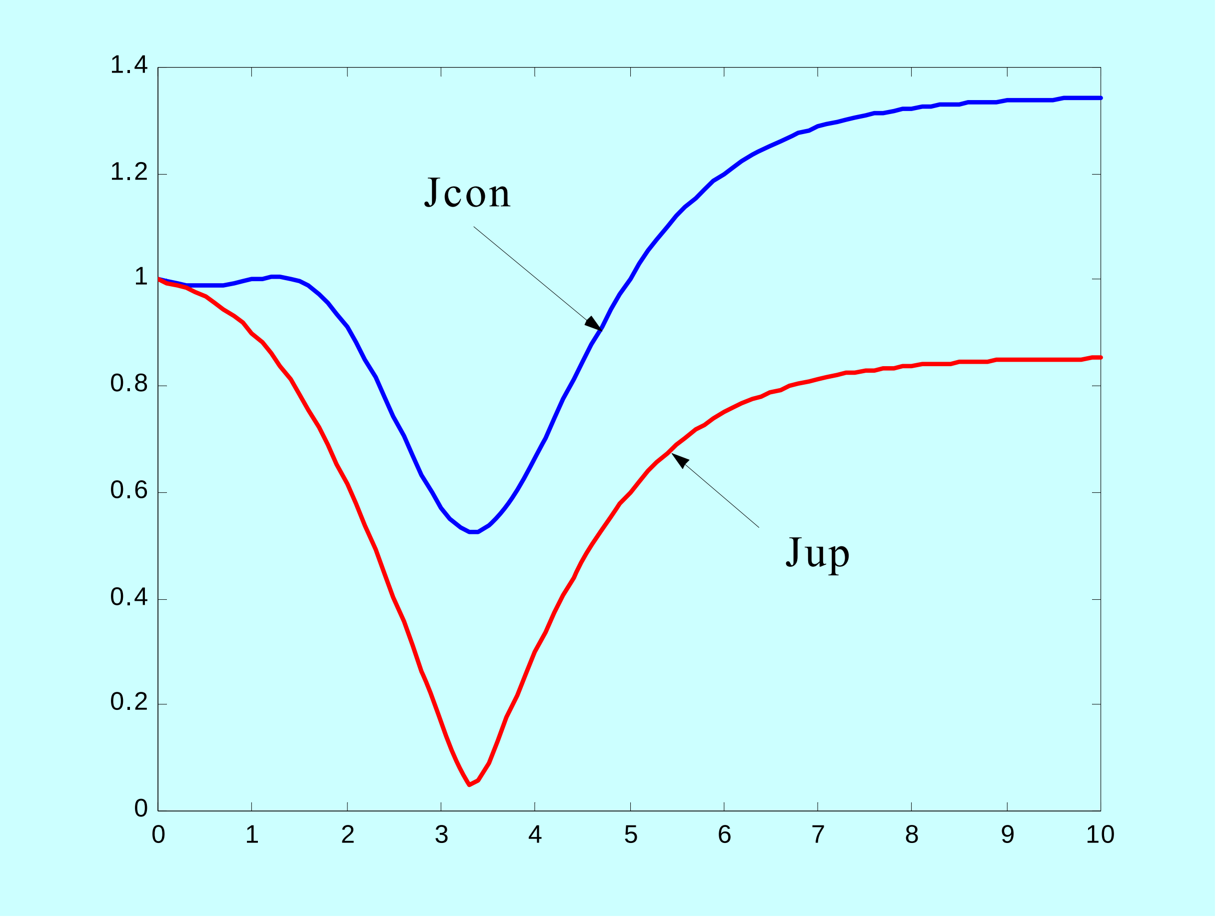
$$J_{con} = \frac{\left\| K - K_{con} \right\|_F}{\left\| K_{con} \right\|_F}$$

Premise: *Unconstrained updated models with smaller connectivity cost are closer to the true model.*

Visualization of the Premise

$$J_{con} = \frac{\left\| K - K_{con} \right\|_F}{\left\| K_{con} \right\|_F}$$

$$\mathrm{J}_{up} = \frac{\left\| K - K_T \right\|_{\mathrm{F}}}{\left\| K_T \right\|_{\mathrm{F}}}$$



The Updating Algorithm

$$J(\beta) = \mathrm{J}_{\mathrm{con}}\left(W = diag\left\{d_i^{\,\beta}\right\}\right)$$

I. Iterate β, and for each value

1) Solve for the unconstrained K(β).

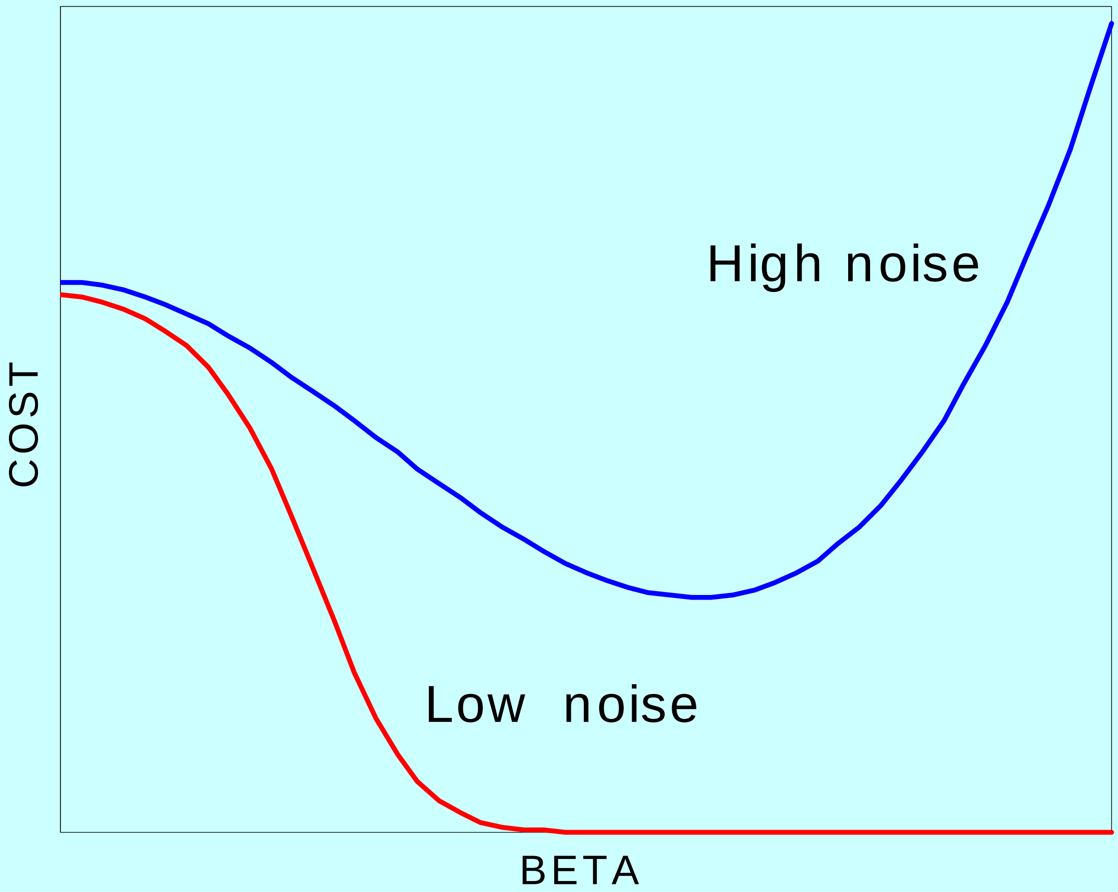
2) Apply connectivity to K(β) and find K_{con}(β).

3) Calculate J(β).

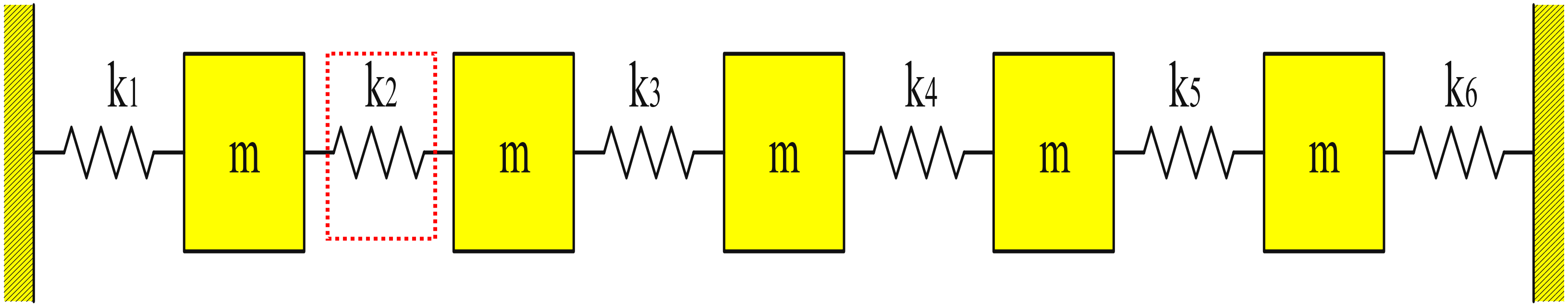
II. Find β* which minimizes J(β).

III. Final tuning (optional): with β= β* repeat I recursively with

$$K_A^{(i)} = K_{con}^{(i-1)}$$


High noise
Low noise
COST
BETA

Example



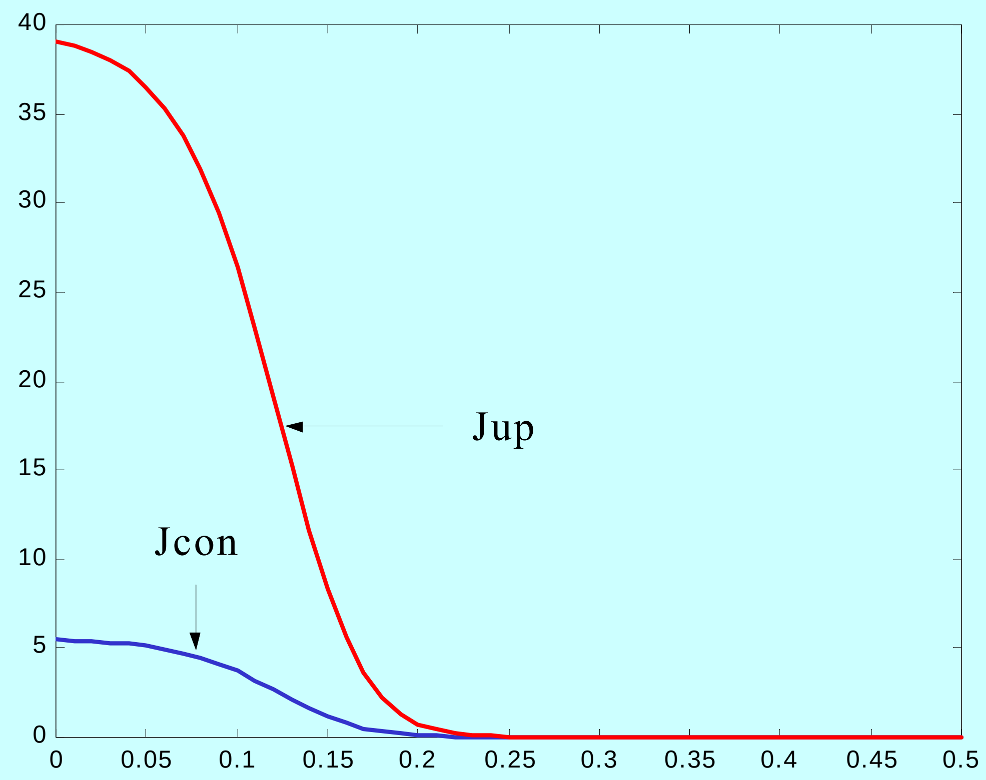
Analytic model : $m=1,\ k_1 = k_2 = 50 = k_3 = k_4 = k_5 = k_6 = 50.$

True model : $m=1,\ k_1 = k_3 = k_4 = k_5 = k_6 = 50,\ k_2 = 70.$

$$K_A = \begin{bmatrix} 100 & -50 & 0 & 0 & 0 \\ -50 & 100 & -50 & 0 & 0 \\ 0 & -50 & 100 & -50 & 0 \\ 0 & 0 & -50 & 100 & -50 \\ 0 & 0 & 0 & -50 & 100 \end{bmatrix} \qquad K_T = \begin{bmatrix} 120 & -70 & 0 & 0 & 0 \\ -70 & 120 & -50 & 0 & 0 \\ 0 & -50 & 100 & -50 & 0 \\ 0 & 0 & -50 & 100 & -50 \\ 0 & 0 & 0 & -50 & 100 \end{bmatrix}$$

Example (cont.)

$$J_{con} = \|K - K_{con}\|_F \qquad\qquad J_{up} = \|K_T - K_{con}\|_F$$



Example (cont.)

β=0

K=

97.8790	-50.6112	**-1.9625**	**-1.7271**	**-1.0051**
-50.6112	103.0060	-48.2799	**1.5138**	**0.8810**
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-1.0051	**0.8810**	**-0.0892**	-50.0785	99.9543

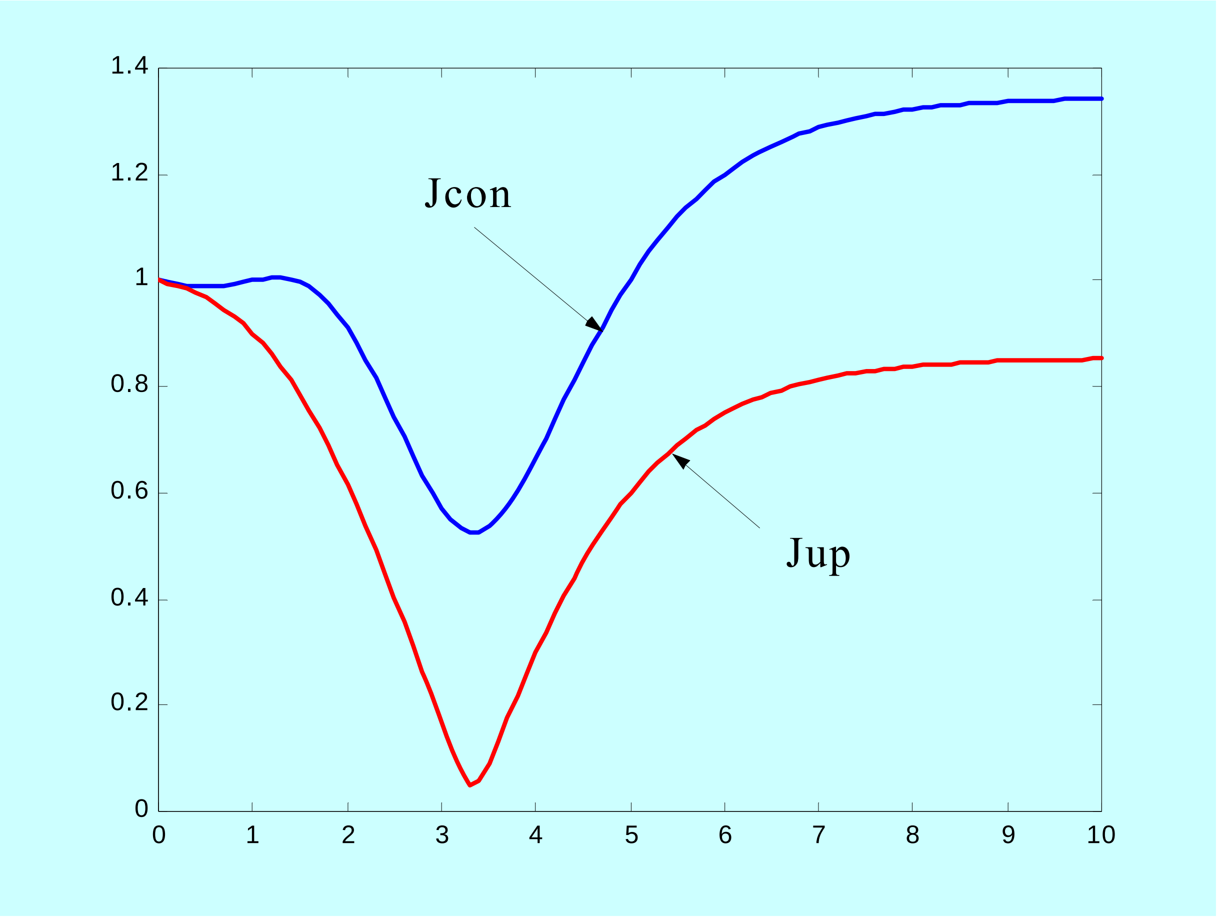
$k_{1\text{-}6}$={46.27, 51.60, 49.40, 50.16, 49.95, 50.00}

β=0.3

K=

120.0000	-70.0000	0.0000	0.0000	0.0000
-70.0000	120. 0000	-50. 0000	0.0000	0.0000
0.0000	-50. 0000	100. 0000	-50. 0000	0.0000
0.0000	0.0000	-50. 0000	100.0000	-50. 0000
0.0000	0.0000	0.0000	-50. 0000	100. 0000

Noisy Measurements



$k_{1\text{-}5}$={49.19, 70.27, 49.05, 50.07, 49.98, 50.03}

Fundamental Question

Consider the two sets

$$S_E = \left\{ K : K\Phi = M\Phi\Omega^2 \right\}$$

$$S_{GRB} = \left\{ K : K = M\Phi\Omega^2\Phi^T M + P\left(K_A - M\Phi\Omega^2\Phi^T M\right)P^T \right\}$$

The projection

$$P = I - W\Phi\left(\Phi^T W\Phi\right)^{-1}\Phi^T$$

Projects into the subspace

$$U = null\left(\Phi^T\right) = span\left(M\tilde{\Phi}\right)$$

Therefore $S_{GRB} \subseteq S_E$. but is $S_{GRB} = S_E$?

Can any K that produces Ω and Φ be obtained by GRB ?

Fundamental Question - refined

Let

$$B = null\left(\Phi^T\right) \quad , \quad B^T B = I_{n-m}$$

Then any $K \in S_{ED}$ has the form

$$K = M\Phi\Omega^2\Phi^T M + M\tilde{\Phi}\tilde{\Omega}^2\tilde{\Phi}^T M = M\Phi\Omega^2\Phi^T M + BXB^T \qquad X = X^T$$

On the other hand

$$K_{GRB} = M\Phi\Omega^2\Phi^T M + P\left(K_A - M\Phi\Omega^2\Phi^T M\right)P^T$$

$$P = I - W\Phi\left(\Phi^T W\Phi\right)^{-1}\Phi^T = BL \qquad (LB = I_{n-m})$$

Question: Is there exist, for any X>0, a projection P (matrix L), so that

$$P\left(K_A - M\Phi\Omega^2\Phi^T M\right)P^T \left(= BL\left(K_A - M\Phi\Omega^2\Phi^T M\right)L^T B^T\right) = BXB^T$$

Fundamental Question - answered

Write
$$K_A = \begin{bmatrix} M\Phi & B \end{bmatrix} \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12}^T & Y_2 \end{bmatrix} \begin{bmatrix} \Phi^T M \\ B^T \end{bmatrix}$$

Then
$$P\left(K_A - M\Phi\Omega^2\Phi^T M\right)P^T = P\begin{bmatrix} M\Phi & B \end{bmatrix} \begin{bmatrix} Y_1 - \Omega^2 & Y_{12} \\ Y_{12}^T & Y_2 \end{bmatrix} \begin{bmatrix} \Phi^T M \\ B^T \end{bmatrix} P^T$$

$$= \begin{bmatrix} PM\Phi & B \end{bmatrix} \begin{bmatrix} Y_1 - \Omega^2 & Y_{12} \\ Y_{12}^T & Y_2 \end{bmatrix} \begin{bmatrix} \Phi^T M P^T \\ B^T \end{bmatrix} = \begin{bmatrix} BZ & B \end{bmatrix} \begin{bmatrix} Y_1 - \Omega^2 & Y_{12} \\ Y_{12}^T & Y_2 \end{bmatrix} \begin{bmatrix} Z^T B^T \\ B^T \end{bmatrix}$$

$$= B\left(Z\left(Y_1 - \Omega^2\right)Z^T + ZY_{12} + Y_{12}^T Z^T + Y_2\right)B^T$$

Question: Is there a **real** solution Z to the equation

$$Z\left(Y_1 - \Omega^2\right)Z^T + ZY_{12} + Y_{12}^T Z^T + Y_2 = X$$

Answer: Not always

Example

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad K_T = \begin{bmatrix} 1+k & -k \\ -k & 1+k \end{bmatrix}, \quad K_A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\boxed{\omega_m = \omega_1 = 1, \quad \phi_m = \phi_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \quad \omega_u = \omega_2 = \sqrt{1+2k}, \quad \phi_u = \phi_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} a+1 & a \end{bmatrix} = \begin{bmatrix} a+1 & a \\ -(a+1) & -a \end{bmatrix}$$

$$K = M\Phi\Omega^2\Phi^T M + P\left(K_A - M\Phi\Omega^2\Phi^T M\right)P^T = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\left[4a^2 + 4a + 5\right]\begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$\min\left[4a^2 + 4a + 5\right] = 4 \quad \Rightarrow \quad \omega_2^2 = 1 + 2k \geq 4 \quad \Rightarrow \quad k \geq \tfrac{3}{2}$$

Conclusion: If $k<3/2$ there is no weight W that leads to the true K_T

Manifolds Distance Minimization (temporary name)

$$\min_{X,\alpha} \quad J = \left\| \left(K_A + \sum_{i=1}^{p} \alpha_i K_i \right) - \left(M\Phi\Omega^2\Phi^T M + BXB^T \right) \right\|_F^2$$

$$\frac{\partial J}{\partial X} = X - \sum_{i=1}^{p} \alpha_i B^T K_i B - B^T \left(K_A - M\Phi\Omega^2\Phi^T M \right) B = 0$$

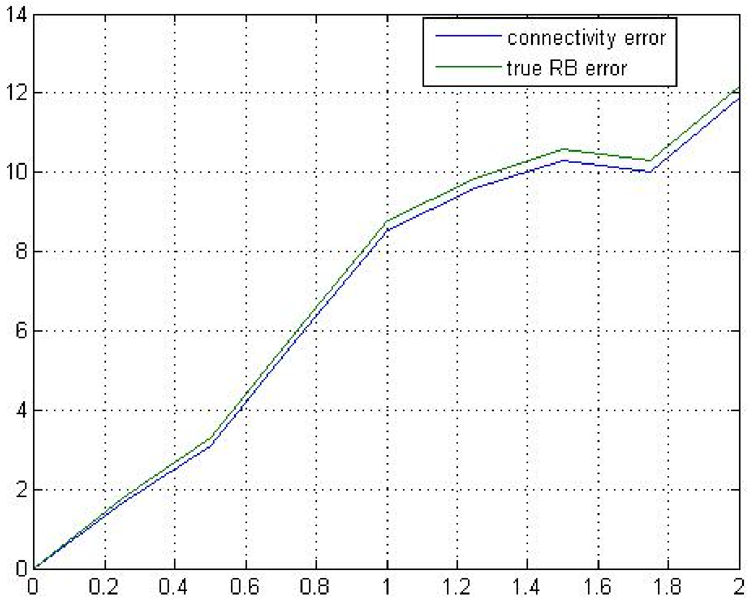
$$\frac{\partial J}{\partial \alpha_i} = trace\left\{ K_i \left(\sum_{i=1}^{p} \alpha_i K_i - BXB^T + \left(K_A - M\Phi\Omega^2\Phi^T M \right) \right) \right\} \quad , \quad i = 1, \ldots, p$$

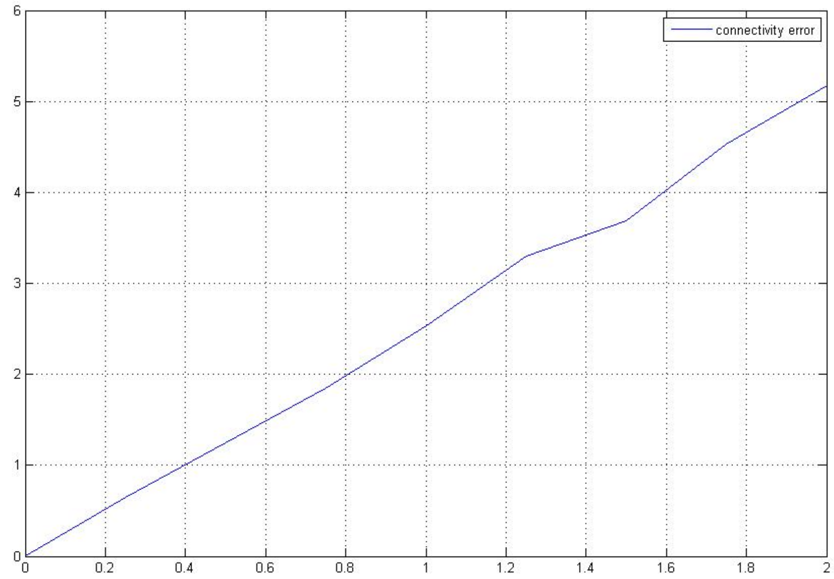
$$P = BB^T = I - \Phi\left(\Phi^T\Phi\right)^{-1}\Phi$$

$$\tilde{K}_i = K_i - PK_iP^T \quad \Rightarrow \quad \tilde{K}_A = \left(K_A - M\Phi\Omega^2\Phi^T M\right) - P\left(K_A - M\Phi\Omega^2\Phi^T M\right)P^T$$

$$\boxed{\begin{bmatrix} tr\left(K_1\tilde{K}_1\right) & \cdots & tr\left(K_1\tilde{K}_p\right) \\ \vdots & \ddots & \vdots \\ tr\left(K_p\tilde{K}_1\right) & \cdots & tr\left(K_p\tilde{K}_p\right) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} -tr\left(K_1\tilde{K}_A\right) \\ \vdots \\ -tr\left(K_p\tilde{K}_A\right) \end{bmatrix}}$$

Manifolds Distance Minimization-Sensitivity to Noise





Connectivity Constrained RB

Manifold Distance Minimization

Summary

- Reference basis with a posteriori connectivity assignment brings together the classical reference basis and parametric methods (e.g. sensitivity).
- It maintains the advantages of the classical reference basis, in particular the low computation effort, and avoids some of its disadvantages.
- The class of system that can be obtained by generalized reference basis does not include all matrices compatible with the eigendata.
- Manifold minimization method is more general than CCRB and seems to be less sensitive to noise.