

Recent Results and Research Problems on Hua Matrices

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Hua, Loo-Keng
Inequalities involving determinants. (Chinese. English summary)
Acta Math. Sinica **5** (1955), 463–470.

Let Z, W be complex-valued $n \times n$ matrices, $\overline{Z}', \overline{W}'$ their conjugate transposes, and let I be the identity matrix. If $I - Z\overline{Z}' > 0, I - W\overline{W}' > 0$, then the following identity holds:

$$(1) \quad d(I - Z\overline{Z}')d(I - W\overline{W}') \leq |d(I - Z\overline{W}')|^2,$$

where d denotes the determinant. More generally, let X_1, \dots, X_m be m square matrices of order n with complex elements, and $\varrho > 0$. If $I - X_i\overline{X}_i' > 0, 1 \leq i \leq m$, then the Hermitian matrix

$$(2) \quad \begin{pmatrix} d(I - X_1\overline{X}_1')^{-\varrho-n+1} \cdots d(I - X_1\overline{X}_m')^{-\varrho-n+1} \\ \vdots \\ d(I - X_m\overline{X}_1')^{-\varrho-n+1} \cdots d(I - X_m\overline{X}_m')^{-\varrho-n+1} \end{pmatrix}$$

is positive semi-definite. The author gives an elementary proof of the first inequality. Proof of its generalization (2) makes use of the representation theory of linear groups.

 Reviewed by *S. Chern*

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Hua 1910-1985



Chern 1911-2004

Two basic properties of contractions:

- A matrix is a contraction iff it is a finite convex combination of unitary matrices;
- If $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0$, then $X = A^{1/2}CB^{1/2}$ for some contraction C .

L.-K. Hua:

Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains

Beijing 1956; Translations Moscow 1958; AMS Translations of Mathematical Monographs, 1963.

Harmonic Analysis, Several Complex Variables, Analysis on Complex Lie Groups and Homogeneous Spaces, Operator Theory...

A recent reference:

Bruce Gilligan and Guy J. Roos: *Symmetries in complex analysis*, Contemp. Math. Vol. 468 (2008).

Result 1: Hua's det inequality

Loo-Keng Hua proved in 1955:

For square strict contraction matrices A and B of the same order

$$|\det(I - A^*B)|^2 \geq \det(I - A^*A) \cdot \det(I - B^*B)$$

↑

$$|\det(I - A^*B)|^2 \geq \det(I - A^*A)\det(I - B^*B) + |\det(A - B)|^2$$

↑

Hua's Matrix Identity:

$$(I - B^*B) + (A - B)^*(I - AA^*)^{-1}(A - B) = (I - B^*A)(I - A^*A)^{-1}(I - A^*B).$$

From determinant to matrix:

$$\begin{pmatrix} \det(I - A^*A)^{-1} & \det(I - B^*A)^{-1} \\ \det(I - A^*B)^{-1} & \det(I - B^*B)^{-1} \end{pmatrix} \geq 0$$

and

$$\begin{pmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{pmatrix} \geq 0.$$

From determinant to matrix:

$$\begin{pmatrix} \det(I - A^*A)^{-1} & \det(I - B^*A)^{-1} \\ \det(I - A^*B)^{-1} & \det(I - B^*B)^{-1} \end{pmatrix} \geq 0$$

and

$$\begin{pmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{pmatrix} \geq 0.$$

How about higher dimensions? And how about

$$\begin{pmatrix} (I - A^*A)^{-1} & (I - A^*B)^{-1} \\ (I - B^*A)^{-1} & (I - B^*B)^{-1} \end{pmatrix} \geq 0?$$

Notice that:

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \not\Rightarrow \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \geq 0.$$

Hua Matrices

A and B are $n \times n$ strict contractions. Form four 2×2 Hua matrices:

$$G \equiv \begin{pmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{pmatrix},$$

$$G_* \equiv \begin{pmatrix} (I - AA^*)^{-1} & (I - BA^*)^{-1} \\ (I - AB^*)^{-1} & (I - BB^*)^{-1} \end{pmatrix},$$

$$H \equiv \begin{pmatrix} (I - A^*A)^{-1} & (I - A^*B)^{-1} \\ (I - B^*A)^{-1} & (I - B^*B)^{-1} \end{pmatrix},$$

$$H_* \equiv \begin{pmatrix} (I - AA^*)^{-1} & (I - AB^*)^{-1} \\ (I - BA^*)^{-1} & (I - BB^*)^{-1} \end{pmatrix}.$$

Hua: $G \geq 0$. Ando: $H \geq 0$. How are G and H related?

Sec. 2 History: Existing Results

- L.-K. Hua, Acta Math. 1955;
- M. Marcus, Monthly 1958;
- R. Bellman, J. Duke Math. 1959;
- T. Ando, LAMA 1980;
- C. Paige, G.P.H. Styan, B.-Y. Wang, F. Zhang, IJISS 2008;
- T. Ando, BJMA, 2008;
- C. Xu, Z. Xu, F. Zhang, LAA 2009;
- G. Xu, C. Xu, F. Zhang, LAMA 2009...

Sec. 3 New Results and Research Problems for 2×2 case

Hua's det inequality is easily generalized to the class of Liebian functions \mathcal{L} having the property $\mathcal{L}(X^{-1}) = (\mathcal{L}(X))^{-1}$:

$$\mathcal{L}(I - A^*A)\mathcal{L}(I - B^*B) \leq |\mathcal{L}(I - A^*B)|^2. \quad (1)$$

Liebian function \mathcal{L} is a positive continuous function from a matrix space to a number field such that

$$\begin{pmatrix} R & S \\ S^* & T \end{pmatrix} \geq 0 \quad \Rightarrow \quad |\mathcal{L}(S)|^2 \leq \mathcal{L}(R)\mathcal{L}(T).$$

As is also known (XXZ LAA 2009), for the trace, which is Liebian but does not have the inverse property,

$$\text{tr}(I - A^*A)\text{tr}(I - B^*B) \leq |\text{tr}(I - A^*B)|^2. \quad (2)$$

Permanent is a Liebian function.

Problem (#1)

Is it true or not that for contractive matrices A and B ,

$$\text{per}(I - A^*A)\text{per}(I - B^*B) \leq |\text{per}(I - A^*B)|^2? \quad (3)$$

Generalization: Let $A, B, C,$ and D be $n \times n$ complex matrices. If

$$A^*A > B^*B \quad \text{and} \quad C^*C > D^*D,$$

then

$$\begin{pmatrix} (A^*A - B^*B)^{-1} & (C^*A - D^*B)^{-1} \\ (A^*C - B^*D)^{-1} & (C^*C - D^*D)^{-1} \end{pmatrix} \geq 0.$$

Thus for any Liebian function \mathcal{L} having property $\mathcal{L}(X^{-1}) = (\mathcal{L}(X))^{-1}$,

$$\mathcal{L}(A^*A - B^*B)\mathcal{L}(C^*C - D^*D) \leq |\mathcal{L}(A^*C - B^*D)|^2.$$

In particular,

$$\det(A^*A - B^*B) \det(C^*C - D^*D) \leq |\det(A^*C - B^*D)|^2.$$

Problem (#2)

Does this hold for trace?

Problem (#3)

Does this hold for permanent?

$$L = \begin{pmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{pmatrix}$$

Observe that when A is strictly contractive, $0 < I - A^*A \leq I$. Since $(I - A^*A)^{-1} \geq I + A^*A$, all the entries on the main diagonal of $(I - A^*A)^{-1}$ are greater than or equal to 1. Let L be any of the Hua matrices G , G_* , H , and H_* .

$$(1, 1, \dots, 1) \leq d(L) \prec (\lambda((I - A^*A)^{-1}), \lambda((I - B^*B)^{-1})) \prec \lambda(L).$$

It is immediate that $\lambda_{\max}(L) \geq 1$.

Problem (# 4)

Do we have $\lambda_{\min}(L) \leq 1$?

Using the identity $(I - XY)^{-1} = I + X(I - YX)^{-1}Y$, we get

$$\begin{aligned}
 H_* &= \begin{pmatrix} (I - AA^*)^{-1} & (I - AB^*)^{-1} \\ (I - BA^*)^{-1} & (I - BB^*)^{-1} \end{pmatrix} \\
 &= \begin{pmatrix} I + A(I - A^*A)^{-1}A^* & I + A(I - B^*A)^{-1}B^* \\ I + B(I - A^*B)^{-1}A^* & I + B(I - B^*B)^{-1}B^* \end{pmatrix} \\
 &= \begin{pmatrix} I & I \\ I & I \end{pmatrix} + (A \oplus B)G(A \oplus B)^* \geq 0.
 \end{aligned}$$

Likewise (or by switching A with A^* , B with B^*)

$$H = \begin{pmatrix} (I - A^*A)^{-1} & (I - A^*B)^{-1} \\ (I - B^*A)^{-1} & (I - B^*B)^{-1} \end{pmatrix} = \begin{pmatrix} I & I \\ I & I \end{pmatrix} + XG_*X^* \geq 0.$$

This is no longer true for three matrices.

Theorem (XXZ, LAMA 2009)

H and H_* each have at least n eigenvalues greater than or equal to 2. H (or H_*) has only eigenvalues 2 and 0 (in duplicates) if and only if $A = B = 0$.

$$H \geq \begin{pmatrix} I & I \\ I & I \end{pmatrix} = J_2 \otimes I.$$

We note here that it is possible that H (and H_*) has n eigenvalues greater than or equal to 2. For instance, when $n = 2$, H (and H_*) may have three eigenvalues strictly greater than 2.

Problem (# 5)

- (a) Is it possible that all eigenvalues of H are greater than 2?
- (b) What if the largest eigenvalue of H is 2?

Note that $G \not\geq \begin{pmatrix} I & I \\ I & I \end{pmatrix}$ in general. Take $A = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = A^*$.

$$G = \begin{pmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{pmatrix} \geq 0.$$

Theorem (XXZ, LAMA 2009)

The largest eigenvalues of G and G_ are greater than or equal to 2.*

Problem (# 6)

- (a) *Can it be that $\lambda_2(G) < 2$?*
- (b) *Does G also have n eigenvalues greater than or equal to 2?*
- (c) *If the largest eigenvalue of H or G is 2, must $A = B = 0$?*

For the closeness of H to G , we have

Theorem (XXZ, LAMA 2009)

Let $\|X\|$ denote the spectral norm of matrix X . Then

$$\|G - H\| \leq \frac{\|B^*A - A^*B\|}{(1 - \|A^*B\|)^2}.$$

Thus

$$\|G - H\| \leq \frac{2\|A^*B\|}{(1 - \|A^*B\|)^2}.$$

Problem (# 7)

Estimate $\|L_1 - L_2\|$, where L_1 and L_2 are any two Hua matrices.

Sec. 4 Hua's positivity Theorem

Hua (1955): For any *real* $k > n - 1$ and *complex* (strict) contractions A 's,

$$D_k = \begin{pmatrix} \frac{1}{\det(I - A_1^* A_1)^k} & \frac{1}{\det(I - A_1^* A_2)^k} & \cdots & \frac{1}{\det(I - A_1^* A_m)^k} \\ \frac{1}{\det(I - A_2^* A_1)^k} & \frac{1}{\det(I - A_2^* A_2)^k} & \cdots & \frac{1}{\det(I - A_2^* A_m)^k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\det(I - A_m^* A_1)^k} & \frac{1}{\det(I - A_m^* A_2)^k} & \cdots & \frac{1}{\det(I - A_m^* A_m)^k} \end{pmatrix} \geq 0. \quad (4)$$

Bellman (1959) partially generalized Hua's (4):

- (a) For *complex* contractions A 's and positive integers k , $D_k \geq 0$;
- (b) For *real* contractions A 's and $k = \frac{p}{2}$, $p = 1, 2, \dots$, $D_k \geq 0$;
- (c) For *real* contractions A 's and real $k < -\frac{1}{2}(n - 1)$, $D_k \geq 0$.

Problem (# 8)

A matrix (or an elementary) proof of either Hua's or Bellman's results?

Problem (# 9)

Let A_1, A_2, \dots, A_m be strictly contractive $p \times q$ complex matrices. Is it true that for all real $k > 0$ (or for what k)

$$D_k = \begin{pmatrix} \frac{1}{\det(I-A_1^*A_1)^k} & \frac{1}{\det(I-A_1^*A_2)^k} & \cdots & \frac{1}{\det(I-A_1^*A_m)^k} \\ \frac{1}{\det(I-A_2^*A_1)^k} & \frac{1}{\det(I-A_2^*A_2)^k} & \cdots & \frac{1}{\det(I-A_2^*A_m)^k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\det(I-A_m^*A_1)^k} & \frac{1}{\det(I-A_m^*A_2)^k} & \cdots & \frac{1}{\det(I-A_m^*A_m)^k} \end{pmatrix} \geq 0?$$

Remarks:

- 1). It involves complex numbers raised to a real power.
- 2). $p = q$ case contains Hua's and (partial) Bellman's results.
- 3). For positive integers k , Bellman's result generalizes Hua's, since

$$\begin{pmatrix} \frac{1}{\det(I-A_1^*A_1)} & \frac{1}{\det(I-A_1^*A_2)} & \cdots & \frac{1}{\det(I-A_1^*A_m)} \\ \frac{1}{\det(I-A_2^*A_1)} & \frac{1}{\det(I-A_2^*A_2)} & \cdots & \frac{1}{\det(I-A_2^*A_m)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\det(I-A_m^*A_1)} & \frac{1}{\det(I-A_m^*A_2)} & \cdots & \frac{1}{\det(I-A_m^*A_m)} \end{pmatrix} \geq 0. \quad (5)$$

Removing the *det*, is the matrix H_m positive semidefinite? where

$$H_m = \begin{pmatrix} (I - A_1^* A_1)^{-1} & (I - A_1^* A_2)^{-1} & \dots & (I - A_1^* A_m)^{-1} \\ (I - A_2^* A_1)^{-1} & (I - A_2^* A_2)^{-1} & \dots & (I - A_2^* A_m)^{-1} \\ \dots & \dots & \dots & \dots \\ (I - A_m^* A_1)^{-1} & (I - A_m^* A_2)^{-1} & \dots & (I - A_m^* A_m)^{-1} \end{pmatrix}, \quad (6)$$

where all A_i are strictly contractive n -square matrices. How about

$$\tilde{H}_m = \begin{pmatrix} (I - A_1^* A_1)^{-1} & (I - A_2^* A_1)^{-1} & \dots & (I - A_m^* A_1)^{-1} \\ (I - A_1^* A_2)^{-1} & (I - A_2^* A_2)^{-1} & \dots & (I - A_m^* A_2)^{-1} \\ \dots & \dots & \dots & \dots \\ (I - A_1^* A_m)^{-1} & (I - A_2^* A_m)^{-1} & \dots & (I - A_m^* A_m)^{-1} \end{pmatrix} ? \quad (7)$$

Neither one is psd in general when $m = 3$.

Theorem (XXZ, LAA 2009)

Let A_1, A_2, \dots, A_m be strictly contractive $p \times q$ matrices. Then

$$T_m = \begin{pmatrix} \frac{1}{\operatorname{tr}(I - A_1^* A_1)} & \frac{1}{\operatorname{tr}(I - A_1^* A_2)} & \cdots & \frac{1}{\operatorname{tr}(I - A_1^* A_m)} \\ \frac{1}{\operatorname{tr}(I - A_2^* A_1)} & \frac{1}{\operatorname{tr}(I - A_2^* A_2)} & \cdots & \frac{1}{\operatorname{tr}(I - A_2^* A_m)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\operatorname{tr}(I - A_m^* A_1)} & \frac{1}{\operatorname{tr}(I - A_m^* A_2)} & \cdots & \frac{1}{\operatorname{tr}(I - A_m^* A_m)} \end{pmatrix} \geq 0.$$

Z (08/09/09): Let $\operatorname{tr}^k X$ denote $(\operatorname{tr} X)^k$, where $k > 0$ is real.

Let A_1, A_2, \dots, A_m be strictly contractive $p \times q$ matrices. Then

$$T_{k,m} = \begin{pmatrix} \frac{1}{\operatorname{tr}^k(I - A_1^* A_1)} & \frac{1}{\operatorname{tr}^k(I - A_1^* A_2)} & \cdots & \frac{1}{\operatorname{tr}^k(I - A_1^* A_m)} \\ \frac{1}{\operatorname{tr}^k(I - A_2^* A_1)} & \frac{1}{\operatorname{tr}^k(I - A_2^* A_2)} & \cdots & \frac{1}{\operatorname{tr}^k(I - A_2^* A_m)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\operatorname{tr}^k(I - A_m^* A_1)} & \frac{1}{\operatorname{tr}^k(I - A_m^* A_2)} & \cdots & \frac{1}{\operatorname{tr}^k(I - A_m^* A_m)} \end{pmatrix} \geq 0.$$

Problem (# 10)

Let A_1, A_2, \dots, A_m be strictly contractive $p \times q$ matrices. Then

$$S_{m,k} \equiv \begin{pmatrix} \frac{1}{s_k(I-A_1^*A_1)} & \frac{1}{s_k(I-A_1^*A_2)} & \cdots & \frac{1}{s_k(I-A_1^*A_m)} \\ \frac{1}{s_k(I-A_2^*A_1)} & \frac{1}{s_k(I-A_2^*A_2)} & \cdots & \frac{1}{s_k(I-A_2^*A_m)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{s_k(I-A_m^*A_1)} & \frac{1}{s_k(I-A_m^*A_2)} & \cdots & \frac{1}{s_k(I-A_m^*A_m)} \end{pmatrix} \geq 0,$$

where s_k denote the k th elementary symmetric functions of eigenvalues.

- Thank you -