

## Research on Differential Equations

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There is only one advantage to getting old, other than the obvious one: it's better than the alternative. What is this advantage ? One is more easily forgiven for philosophizing. With that in mind I hope the reader will forgive me for opening my essay with some personal remarks.

My sixty year odyssey from Gakowa, a rural village in Yugoslavia (which was the site of concentration camp Gakowa from 1944-1949), where I was born and raised, to DeKalb, Il. and the writing of this essay, has been eventful. In the fall of 1944 when the war in Europe was essentially over, at age 9, after my third year of grade school, my family and all other members of my ethnic group living in the former Yugoslavia, were interred in concentration camp Gakowa and half a dozen other such camps for the crime of having had ethnic German ancestors. During the next three years the struggle to survive was intense; the danger of dying of starvation or disease ever present and all consuming. Such matters as the education of children was of concern to no one.

In 1947 we escaped and made our way, on foot, through Hungary and the Soviet zone of Austria ending up, after a couple of refugee camps, in Villach, which was in the British zone. There I was able to resume my education by going to fourth grade, after a three year absence, at the age of 12.

At age 16, weighin 98 pounds and standing 5'5", my family and I arrived in Chicago on July 4-th from an internment on Ellis Island, literally penniless. (My first exposure to the U.S. was on Ellis Island where I looked at the Statue of Liberty through a barbed wire fence every day.) Since I was the only member of my family who spoke any English at all - a few words I had learned in Villach - it was my responsibility to telephone friends who had agreed to pick us up at the train station. Since I had never seen a telephone before I had no idea how to use this gadget and waited next to one until some kindhearted person somehow must have figured out what was going on and made the call for me.

After a fight with the education establishment, who wanted to send me to 8-th grade, we managed to get me admitted to Chicago Vocational High School on a provisional basis. I was later allowed to stay, against the instructions of the education establishment from Washington, and graduated from Chicago Vocational High School at age 20.

My goal in College was to become a high school teacher. During my

senior year two of my Professors, Bill Mahavier and John Neuberger, suggested that I continue my studies of mathematics under their guidance in graduate school. The idea of going to graduate school had previously not occurred to me, but upon reflection I decided to do it. That is how I became a mathematician.

### Mathematics Research

This is only the second essay on mathematics in the PRP series (a rather startling statistic in view of the large number of NIU mathematicians with major international reputations). The first was written by Martin Lorenz, an algebraist at Temple University, who had spent two years at NIU before going to Temple. So I want to start by saying a few words about research in Mathematics generally. Describing research in mathematics to an audience not familiar with the language of the subject is a formidable task, one that may well be beyond my capabilities. Nevertheless I will make an attempt.

It is more difficult to explain, to a general audience, research in mathematics than it is to explain research in many other fields, including the Sciences. Not because mathematics is deeper than the sciences, but because scientists have the advantage of being able to build on common human experiences and to rely on intuitive meanings of commonly used terms. Thus a Physicist can talk about an atom and all of us prescribe *some* meaning to this term. She can then go on to tell us what it really is and describe its fascinating properties. Similarly a geologist can talk about ancient volcanic rocks in lake superior and a biologist can discuss the genetic and molecular analysis of certain viruses.

A mathematician does not enjoy such a luxury. People outside the field do not normally encounter the spectrum of a differential operator and even if they did they probably wouldn't be aware of it.

Everything in mathematics, even the numbers 1,2,3, ... has been painstakingly and rigorously defined in terms of basic set theory. So when a mathematician discusses the differential equation

$$y'' + q(t)y = 0 \tag{1}$$

every other mathematician knows precisely what he is talking about. There is no ambiguity, no room for different "interpretations".

Research in mathematics consists of proving theorems. A theorem has two parts: (i) a statement of a fact and (ii) its proof. The statement must be clear and precise, the proof completely rigorous and based entirely on

logic. All terms used in the statement must be painstakingly and accurately defined, the proof cannot leave any doubt about the validity of the statement. Here mathematics differs from the sciences. Trying to prove a theorem merely by confirming its validity in 27,650 special cases is totally unacceptable in mathematics; no graduate student survives a basic course with such an attempt. Some 5,000,000 special cases out of infinitely many possibilities is an insignificant number.

Once a theorem has been proven it becomes part of the heritage of the human race forever. The Pythagorean Theorem established by the Greeks more than 2,000 years ago is as valid (and as useful) today as it was then; the Chinese Remainder Theorem is more than 3,000 years old. A research Mathematician simply adds to the collection of theorems. The number of known theorems increases every day. The total accumulation of theorems is so large that no human knows even one percent of them. Only God, and possibly someone somewhere in the Andromeda galaxy, knows them all and also those yet to be discovered.

How does one discover a theorem? Often part (i) expresses a pattern observed in numerous examples. For part (ii) the challenge then is to *prove* that, given the conditions of (i), this pattern *always* prevails. There are two extremes: On the one hand you can take a known conjecture - a pattern observed by someone else - and prove it. A prime example of this is Andrew Wiles' recent proof of "Fermat's Last Theorem" ( a misnomer - it should have been called Fermat's Last Conjecture). A French lawyer named Fermat, whose hobby was mathematics, claimed, in the margin of a book he was reading at the time, to have proved a theorem. But only part (i) survived him because he died before he was able to publish part (ii). This occurred more than 350 years ago. Fermat's part (i) became so famous that many people devoted their life to finding part (ii) but only Wiles succeeded. If his proof holds up - it is still being checked by experts - Wiles will join Euclid, Archimedes, Newton, Leibnitz, Gauss, Fermat, Euler, etc. as one of the immortals of Mathematics.

At the other extreme are people who find a new pattern for part (i) and succeed in proving it. This way of discovering theorems can be utterly worthless, if no one else is interested, or it can be a major breakthrough. Often asking a good question is more important, and requires more insight, than finding the answer.

### **Differential Equations**

These come in two flavors : ordinary and partial. The former involves func-

tions of one variable, the latter two or more variables. The characteristic feature of any differential equation (de) is the presence of at least one derivative.

What is a derivative ? Those of you who have enjoyed a Calculus course know the answer, those readers who have not benefitted from such a course need to take one to find out what a derivative is. It has a precise meaning which is too technical to be included here; but you can think of it intuitively as a “rate of change”. Everything in the universe, including you and I, changes with time. If we want to study such changes seriously we need to consider the rate of change - hence derivatives get into the act. This means that derivatives, and therefore differential equations, tend to be present in almost every serious technical study of virtually any phenomenon. They play an important role in engineering, science and, of course, mathematics.

One of my research areas is linear ordinary differential equations (ode's) - the others are inequalities involving derivatives and Scientific Computing. My remarks will be restricted to ode's. The classical form of these is the following :

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \dots + p_1 y' + p_0 y = f. \quad (2)$$

Here the  $p$ 's,  $y$ 's and  $f$  are functions of, say  $t$ , and you may think of  $t$  as representing time but it can represent any variable;  $n$  is a positive integer and  $y^{(j)}$  is the  $j$ -th derivative of  $y$ .

Regarding equation (2) my research interest is to determine properties of its solutions  $y$  from the properties of the given coefficients  $p_j, f$ . My ultimate goal is simply to know all there is to know about equation (2) so that if you tell me what the functions  $p_j$  and  $f$  are, I can answer any question you care to ask about the solutions. Are they bounded? periodic? oscillatory? “big”?, “small”?, how big?, how small?, do they satisfy certain boundary conditions ?, etc. There is no chance whatsoever that I or anyone else will achieve this goal in the next 1,000 years.

The case  $n = 1$  is “solved.” For this case there exists a formula to obtain the solutions  $y$  from the coefficients  $p_0, p_1, f$ . My Math 336 students know this formula. Using it, and a knowledge of the elementary functions studied in Calculus *and their integrals* one can answer just about any question you care to ask about equation (2) with  $n = 1$ . It has been *proven* that no such formulas exist for  $n = 2, 3, 4, \dots$

Equation (1) is the special case  $n = 2$  of (2) further simplified by choosing  $p_1(t) = 1$ . Even its solutions cannot be given in “closed form.”

**Why study equation (1) and who cares ?**

Let me try to provide some context for the study of (1). In the 1830's Sturm and Liouville, two French Mathematicians/Physicists (there was no clear distinction between these fields in those days) ran across (1) in their study of the transmission of heat through rods. Their work had a profound influence on Mathematics, Engineering, Physics, and other fields. I will restrict my remarks to Mathematics. They appear to have been the first to realize that (1) cannot be solved in closed form. Consequently the bag of all elementary functions and their integrals is not big enough to hold the solutions of (1). In fact the very concept of "function" which is fundamental in all of mathematics and Science underwent a revolution as a result of their work. It came to be realized that in order to study (1) it is not sufficient just to manipulate formulas; anyone seriously interested in knowing properties of solutions of (1) cannot avoid deriving these properties directly from the equation.

Who cares about (1)? It is mind-boggling to find that this simple *de* which has been intensely studied, not only by Mathematicians but also by Physicists, Engineers, Scientists and others since 1836 is still today an extremely active field of research. As a mathematician my reason for studying (1) is because it's interesting - because "its there." The fact that this equation is of interest to so many people outside of Mathematics is secondary - a nice bonus. A few years ago while I was affiliated with the Mathematics and Computer Science division of Argonne National Laboratory, I did a computer search of the literature going back 10 years for just one aspect of (1) - the so called Sturm-Liouville theory - and found, to my surprise, that as many non mathematicians as mathematicians published papers on (1). Of the 200 papers about 100 were by mathematicians and 100 by others. One of these other papers studied the special case when  $q(t) = 1/t$  for  $-1 \leq t \leq 1$ ,  $t \neq 0$ ; it was authored by an atmospheric scientist who was using it to study eddies in the atmosphere. I noticed that the questions he asked were not answered by the known Mathematics for (1). In several papers with various co-authors we established a theory which answered his questions and much more. With W. N. Everitt, a British collaborator and friend, and a few other co-authors we are working on obtaining some further results for these kinds of problems.

To avoid misleading the reader I want to stress that my motivation for the research problems I work on (the example just discussed notwithstanding) comes from within mathematics - not from outside my discipline. It so happens that I work in an area of mathematics which is a meeting ground for Classical Analysis, Modern Analysis including Functional Analysis, Sci-

entific Computing, Theoretical Physics, especially Quantum Mechanics, and to a lesser extent some specialized subfields of Chemistry and Engineering.

What is the spectrum of the differential operators associated with (1)? This is a question of major interest in Mathematics and Quantum Mechanics. For the latter, (1) is the one dimensional Schrödinger equation. This is the famous equation which models the behavior of elementary particles in Quantum Mechanics. The special case  $q(t) = c/t$  where  $c$  is a constant models the hydrogen atom. The eigenvalues give the possible energy states of the electrons, the eigenfunctions have to do with the probabilities involved in locating the electrons in their “electron cloud”.

With my colleagues Paul Bailey and Norrie Everitt, and with NSF support, we are in the process of finalizing a software code for the numerical computation of the spectrum of differential operators associated with (1) and its main generalizations. By combining this code with some new theoretical results we can find numerical approximations not only of eigenvalues and eigenfunctions but also of continuous spectrum. This code successfully computes the spectrum of the hydrogen atom, but this has been done in other ways. Our code handles many other cases which were previously inaccessible.

It is my hope that young people reading this essay are convinced that in this country, with all its faults, the opportunities are so numerous that anyone can succeed if they are willing to work sufficiently hard. If there is one word to describe my career it is “persistence”. I believe this is the most important trait of a research mathematician and, I suspect, also of successful researchers in many other fields.

Besides the US, I know of no country where a refugee family, utterly penniless, without a home country, with parents whose education stopped with sixth grade, who couldn't speak the language, who didn't know the culture, who had no job skills, would have the opportunities we had. Thank you, United States of America!