Steps for Partial Fraction Decomposition

Example:
\[
\frac{2x^5 + 4x^4 + x^3 + 4x^2 + x}{x^6 + x^5 + x^4 + x^3}
\]

1. **Do any long division needed and cancel any common factors of the numerator and denominator.** For our example there is no long division to do (the degree of the numerator is smaller than the degree of the denominator), but we can cancel an \(x\) to get
\[
\frac{2x^4 + 4x^3 + x^2 + 4x + 1}{x^5 + x^4 + x^3 + x^2}.
\]

2. **Completely factor the denominator into a product of powers of distinct linear and irreducible quadratic terms.** For our example
\[
x^5 + x^4 + x^3 + x^2 = x^2(x^3 + x^2 + x + 1) = x^2(x + 1)(x^2 + 1).
\]
The \(x^2\) (you may want to think of it as \((x - 0)^2\)) is the square of a linear term, \(x + 1\) is another linear term, and \(x^2 + 1\) is an irreducible quadratic term.

3. **For each linear term, write**
\[
\frac{A_i}{\text{linear term}^\text{power}}
\]
and repeat this (changing the subscript and the power) until you reach the power of the linear term in the factorization. For each irreducible quadratic term, write
\[
\frac{A_ix + B_j}{\text{quadratic term}^\text{power}}
\]
until you reach the power of the quadratic term in the factorization. The \(A_i\)'s and \(B_j\)'s represent numbers you are to find. For our example we write
\[
\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x + 1} + \frac{A_4x + B_1}{x^2 + 1}.
\]

4. **Add the fractions above using a common denominator and group by powers of \(x\) in the numerator.** For our example
\[
\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x + 1} + \frac{A_4x + B_1}{x^2 + 1}
\]
\[
= \frac{A_1(x)(x + 1)(x^2 + 1) + A_2(x + 1)(x^2 + 1) + A_3(x^2)(x^2 + 1) + (A_4x + B_1)(x^2)(x + 1)}{x^5 + x^4 + x^3 + x^2}
\]
\[
= \frac{A_1(x^4 + x^3 + x^2 + x) + A_2(x^3 + x^2 + x + 1) + A_3(x^4 + x^2) + (A_4x + B_1)(x^3 + x^2)}{x^5 + x^4 + x^3 + x^2}
\]
\[
= \frac{x^4(A_1 + A_2 + A_3 + A_4) + x^3(A_1 + A_2 + A_4 + B_1) + x^2(A_1 + A_2 + A_3 + B_1) + x(A_1 + A_2) + A_2}{x^5 + x^4 + x^3 + x^2}
\]

5. Equate the coefficients of the powers of $x$ above with those in the numerator and solve for the $A_i$’s and $B_j$’s. For our example we have

- $2 = A_1 + A_3 + A_4$ (coefficient of $x^4$)
- $4 = A_1 + A_2 + A_4 + B_1$ (coefficient of $x^3$)
- $1 = A_1 + A_2 + A_3 + B_1$ (coefficient of $x^2$)
- $4 = A_1 + A_2$ (coefficient of $x$)
- $1 = A_2$ (coefficient of $x^0$, i.e, constant term)

From the last line

- $A_2 = 1$

Putting this in the next gives $4 = A_1 + 1$, so

- $A_1 = 3$

Putting these two values into the first three equations gives

- $2 = 3 + A_4 + A_4$
- $4 = 4 + A_4 + B_1$
- $1 = 4 + A_3 + B_1$.

Subtracting the third equation here from the second gives

- $3 = A_4 - A_3$.

You can now take this equation together with the first equation above, $2 = 3 + A_3 + A_4$, and solve for $A_3$ and $A_4$. You get

- $A_3 = -2$, $A_4 = 1$. 

Finally, you can use these values in any of the equations above involving $B_1$ to find $B_1$. You get

- $B_1 = -1$.

6. Write out your answer (and check it). For our example

$$\frac{2x^4 + 4x^3 + x^2 + 4x + 1}{x^3 + x^4 + x^3 + x^2} = \frac{3}{x} + \frac{1}{x^2} - \frac{2}{x + 1} + \frac{x - 1}{x^2 + 1}.$$