The Deflection of a Light Ray by the Sun

This project investigates how a ray of light from a distant source may be curved by the gravitational field of a large body (say a star). For simplicity, we assume that the mass of the body is concentrated at a point $S$, and that the light ray would travel the infinite straight line $\ell$ if there were no deflection. Let $r$ be the shortest distance from $S$ to $\ell$:

![Diagram of light ray deflection]

We assume that the ray is bent into a curve which has two asymptotes, $\ell$ and $a$, which meet in an angle $\alpha$ as shown.

We seek an equation which will give $\alpha$ as a function of the other relevant variables. Two of these variables are clearly the mass $m$ of the body, and the distance $r$.

1. Show that if the method of dimensional analysis is applied to the set of variables $\alpha, m, r$, then no reasonable solution is produced. Conclude that some essential variables must have been left out.

2. Since $\alpha, m, r$ are the only apparent physical variables, the other variables might be physical constants. Two likely candidates are $c$, the speed of light, and $G$, the gravitation constant which appears in Newton’s Law of Universal Gravitation: $F = G \frac{m_1 m_2}{l^2}$, where $F$ is gravitational force between two bodies, $m_1$ and $m_2$ are their masses, and $l$ is the distance between them.
   
   (a) Find the dimensions of $c$ and of $G$.
   
   (b) Apply dimensional analysis to each of the sets $\{\alpha, m, r, c\}$ and $\{\alpha, m, r, G\}$. What can you conclude?

3. Apply the method of dimensional analysis to the set of variables $\alpha, m, r, G$ and $c$, and derive

   $\alpha = \psi \left( \frac{G m}{c^2 r} \right)$  \hspace{1cm} (I)

   where $\psi$ is an unknown function.

4. To proceed from (I), and derive a specific formula (or at least an approximation) for $\psi$, we need to make some observations, and a couple of assumptions.
(a) Argue from the physical systems being described why each of the following should be true:

i. \( \psi(x) \) is defined for all real \( x \geq 0 \), and the range of \( \psi \) is \([0, \pi/2)\).

ii. \( \psi(x) \) increases as \( x \) increases

iii. \( \lim_{x \to 0} \psi(x) = 0 = \psi(0) \) (Hint: what if \( m \) is very small, or \( r \) is very large?)

(b) Assume that \( \psi(x) \) is differentiable, for all \( x \geq 0 \). Explain why this is reasonable, and why it must follow that \( \frac{d}{dx}(\psi(x)) \geq 0 \) for all \( x \geq 0 \).

(c) Assume that
\[
\lim_{x \to 0} \frac{d}{dx}(\psi(x)) = J = \psi'(0), \quad J \text{ a positive number.} \tag{II}
\]

i. If \( \lim_{x \to 0} \frac{d}{dx}(\psi(x)) \) is going to exist at all, what other possibilities are there?

ii. Given assumption (II), give an argument to explain why, for large \( r \), we have
\[
\alpha \approx J \frac{Gm}{c^2r} \tag{III}
\]

(Hint: consider the graph of \( \psi(x) \) for small \( x \).)

(d) In gram/cm/sec/ units, \( G \approx 7 \times 10^{-11} m^3kg^{-1}s^{-2} \), \( c \approx 3 \times 10^8 m/s \), and the mass of the sun is \( m \approx 2 \times 10^{30} kg \). Suppose for an observed ray of starlight passing by the sun, \( r \approx 7 \times 10^8 m \) and \( \alpha \approx 1.7 \) seconds of arc. Compute \( \frac{Gm}{c^2r} \) and apply (III) to compute \( J \).

(e) Suppose a light ray passes by a massive galaxy that is \( 10^{12} \) times the mass of the sun at a distance of \( 2.1 \times 10^{21} m \). How large will the deflection be? Express the answer in arc seconds.