Ph.D. Qualifying Examination A
Algebra
June 2016

Instructions: For the two-hour examination, work **Part A** only. For the three-hour examination, work **Part A** and **Part B**.

Part A Solve seven of the following eight problems.

1. Let G be a group, and let H and K be isomorphic subgroups of G. Prove or disprove: *If H is normal in G, then K is normal in G.*

2. Prove that there is no simple group of order 36.

3. Let p be a prime number, and let G be a nontrivial finite p-group.
   (a) Let X be a finite set on which G acts. Prove that $|X| \equiv |X^G| \pmod{p}$, where $X^G = \{x \in X \mid a \cdot x = x \text{ for all } a \in G\}$.
   (b) Use part (a) to deduce that the center of G is nontrivial.

4. Consider the ring $R = \mathrm{M}_2(\mathbb{Z})$ consisting of all 2×2 matrices with integer entries. Let I be the subset of R consisting of all matrices with even entries. Then I is an ideal of R and the factor ring R/I is finite. **Your task:** Determine the exact number of elements contained in R/I.

5. Let F/K be a field extension, and suppose that $u \in F$ is algebraic over K.
   (a) Prove that there exists a unique monic irreducible polynomial $p(x) \in K[x]$ with $p(u) = 0$.
   (b) Prove that if $f(x)$ is any polynomial in $K[x]$ with $f(u) = 0$, then $p(x)$ divides $f(x)$ in $K[x]$.

6. Determine the isomorphism type of the Galois group of the polynomial $f(x) = x^4 - 2$ over $\mathbb{Q}(i)$.

7. Let F/K be a Galois extension, and suppose that $\mathrm{Aut}(F/K) \cong S_4$.
   (a) Prove that there are at least 10 distinct fields strictly between K and F.
   (b) Prove that there is a field E strictly between F and K such that E/K is Galois. Describe the Galois group of E/K.

8. Let V be a finite-dimensional complex vector space, and let S, T be linear operators on V such that $ST = TS$. Recall that a subspace W of V is said to be invariant under T if $T(W) \subseteq W$.
   (a) Prove that if λ is an eigenvalue of S, then the eigenspace $V_\lambda = \{\vec{x} \in V \mid S(\vec{x}) = \lambda\vec{x}\}$ is invariant under T.
   (b) Prove that S and T have at least one common eigenvector (not necessarily associated to the same eigenvalue).

Part B Solve three of the following four problems.

1. Prove that, in a principal ideal domain, every nonzero prime ideal is maximal.

2. Let R be a ring, and let M be an R-module. Prove that M is artinian (that is, M satisfies the descending chain condition) if and only if every nonempty set of R-submodules of M has a minimal element.

3. Let R be a commutative ring. Prove that R is semisimple if and only if R is isomorphic to a finite direct product of fields.

4. Let T be an integral ring extension of S, and let S be an integral ring extension of R. Prove that T is an integral ring extension of R.