MS Qualifying Examination B

Analysis

January 2016

Instructions: Write your answers to problems A1-A5, your answers to Problems B1-B5, and your answers to problems B6-B10 in separate blue books. All candidates should attempt 4 of the 5 problems in part A. Those taking the two hour examination should work 4 of the 10 problems in Part B. Those taking the three hour examination should work 8 of the 10 problems in part B. **Clearly indicate which problems you wish to have scored.**

Part A: Work 4 of the following 5 problems. Clearly indicate which problem is not to be graded.

A1. Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences.

(a) If $a_n \to 0$ and $(b_n)_{n=1}^{\infty}$ is bounded, prove that the sequence defined by $c_n = a_n b_n$, for $n \in \mathbb{N}$, converges to 0 as $n \to \infty$.

(b) Hence, or otherwise, find the limit of the sequence given by $d_n = \dfrac{n + 2\cos(n)}{3\sin(n) + 4n}$.

A2. Consider the function $f(x) = \begin{cases} x^2 + x^3 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$.

(a) Determine where f is continuous.

(b) Determine where f is differentiable.

Fully explain your answers.

A3. For $n \in \mathbb{N}$ and $n \geq 2$, consider the functions $f_n(x) = \begin{cases} 1 & \text{if } x \in [0, 1/n^2] \\ \dfrac{n^2 x - n}{1 - n} & \text{if } x \in (1/n^2, 1/n) \\ 0 & \text{if } x \in [1/n, 1] \end{cases}$.

(a) Sketch the graph of $y = f_n(x)$.

(b) Prove that the sequence $(f_n)_{n=1}^{\infty}$ converges pointwise on $[0, 1]$ and find the limit.

(c) Prove that $(f_n)_{n=1}^{\infty}$ does not converge uniformly on $[0, 1]$.

A4. Consider series $\sum_{n=1}^{\infty} a_n$ of non-negative terms.

(a) State the Comparison Test and the Ratio Test for such series.

(b) Using these two tests, determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty} \frac{e^n - n^2/10}{(n+1)! \log(1 + 2n)}$.

A5. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a function.

(a) Define what it means for F to be differentiable at $X_0 \in \mathbb{R}^n$.

(b) Find $F'(X_0)$ for

$$F(x, y, z) = \begin{pmatrix} \exp(x^2 - y + \sin(z)) \\ ye^x - 2z^2 \end{pmatrix}, \quad X_0 = (1, 2, \pi/2).$$

(c) Hence write down the affine map that well approximates F near X_0.

Part B.

B1. (a) State the Lebesgue Convergence Theorem.

(b) Find $\lim_{n\to\infty}\int_1^\infty \frac{2n}{e^{2x}+n^2x}dx$ and justify your calculations.

B2. Let E be a bounded set of real numbers.

(a) Prove that there is a G_δ set G such that $E \subset G$, and $m(G) = m^*(E)$.

(b) Let G be a G_δ set in (a). If $m^*(G\backslash E) = 0$, prove that E is measurable.

B3. Prove that f is of bounded variation on $[a, b]$ if and only if there are increasing function g and h on $[a, b]$ such that $f = g - h$.

B4. Let $E \subset [a, b]$ be a measurable set with $m(E) > 0$.

(a) For $a \le x \le b$, define $h(x) = m(E \cap [a, x])$. Prove that h is a continuous function on $[a, b]$.

(b) Prove that $\lim_{h\to 0}\frac{m(E\cap[x,x+h])}{h} = \chi_E(x)$ a.e. on $[a, b]$.

B5. Let $< f_n >$ be a sequence of integrable functions such that $f_n \to f$ with f integrable. Suppose that $\lim_{n\to\infty}\int |f_n(x)|dx = \int |f(x)|dx$. Prove that

$$\lim_{n\to\infty}\int |f(x) - f_n(x)|dx = 0.$$

B6. Let $G = \{z : |z| < 1 \text{ and } \mathrm{Re}z > 0\}$, $H = \{z : \mathrm{Im}z > 0 \text{ and } \mathrm{Re}z > 0\}$ and $S = \{z : |z| < 1\}$.

(a) Find a conformal mapping from G onto H.

(b) Find a conformal mapping from H onto S.

B7. Evaluate $\displaystyle\int_\Gamma \frac{1}{(z+4)^3(z-2i)}dz$, where:

a) Γ is the circle $|z| = 1$ traversed once counterclockwise.

b) Γ is the circle $|z| = 3$ traversed once counterclockwise.

c) Γ is the circle $|z| = 5$ traversed once counterclockwise.

B8. Let f be an entire function in $\mathbb{C}$. If $f(z)$ is real when $|z| = 1$, prove that f is constant in $\mathbb{C}$.

B9. (a) State one of any version of Residue Theorem.

(b) Let f be an entire function and let $a_1, ..., a_n$ be all zeros of f in $\mathbb{C}$. Suppose that there exist real numbers $r_0 > 0$ and $t > 1$ such that $|f(z)| \geq |z|^t$ for all $|z| \geq r_0$. Prove that

$$\sum_{j=1}^{n} \mathrm{Res}(\frac{1}{f},\ a_j) = 0.$$

B10. (a) State one of any version of Rouche's Theorem.

(b) Determine the number of solutions of the equation $z^{10} + 10z + 8 = 0$ in the unit disk $|z| < 1$.