Ph.D. Qualifying Examination D
Differential Equations
June 2015

Instructions: In this three-hour examination, Part A and Part B carry equal weight in determining your overall performance. Please use separate blue books for Part A and Part B.

Answer all 4 questions in Part A and 4/5 questions in Part B.

$\mathcal{PART\ A}$

A1. (a) Determine e^{At} if

$$A = \begin{bmatrix} -1 & 0 & -3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}. \tag{1}$$

(b) Use this to solve the associated nonhomogeneous problem $\dot{x} = Ax + f(t)$ where A is defined in Eq. (1) and $\left[f(t) = \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}. \right]$

A2. Consider the system $\dot{x} = Ax - r^2 x$ where A is a constant $n \times n$ matrix with complex eigenvalues $\alpha_i + i\omega_i$, $i = 1, \ldots, n$, $\alpha_i, \omega_i \in \mathbb{R}$, and $r^2 = |x|^2 = \sum_{i=1}^{n} x_i^2$.

(a) Prove the origin is asymptotically stable if $\alpha_i < 0$, for all $i = 1, \ldots, n$ and unstable if $\alpha_i > 0$ for all $i = 1, \ldots, n$.

(b) If $n = 2$, prove there exists at least one limit cycle if $\alpha_i > 0$, for $i = 1, 2$.

A3. Suppose $t \mapsto x(t)$, and $t \mapsto y(t)$ are solutions of the smooth differential equation $\dot{x} = f(x)$, and are both defined on the interval $[0, T]$. Use Gronwall's inequality to prove there exists an $L > 0$ such that $\left[|x(t) - y(t)| \le |x(0) - y(0)| e^{Lt}. \right]$

A4. Consider the time-periodic system

$$\dot{x} = A(t)x + b(t), \quad x \in \mathrm{R}^n \tag{2}$$

where $t \mapsto A(t)$ is a T-periodic matrix valued function and $t \mapsto b(t)$ is a T-periodic vector valued function. Use the variation of constants formula and Floquet's theorem to prove that if the number one is not a characteristic multiplier of the T-periodic homogeneous system $\dot{x} = A(t)x$, then Eq. (2) has at least one T-periodic solution.

$\mathcal{PART\ B}$

B1. Consider the Poission equation wtih Dirichlet boundary condition

$$\begin{cases} \Delta u = f & \text{in} \quad B(0,R) \\ u = g & \text{on} \quad \partial B(0,R) \end{cases}$$

Where $B(0,R)$ is the ball centered at the origin with radius $R > 0$ in $\mathbb{R}^N$ with $N \geq 3$.
Prove the mean value formula

$$u(0) = \frac{1}{\omega_N R^{N-1}} \int_{\partial B(0,R)} g\, ds + \frac{1}{(N-2)\omega_N} \int_{B(0,R)} (\frac{1}{R^{N-2}} - \frac{1}{1x1^{N-2}}) f dx$$

ω_N being the surface area of the N-dimensional unit sphere.

B2. (i) Let Ω be any domain in $\mathbb{R}^N$, $N \geq 1$ (i.e. an open connected set) and let $f \in L^1_{loc}(\Omega)$.
Define the weak derivatives $\frac{\partial f}{\partial xi}$, $1 \leq i \leq N$.

(ii) Take Ω = (-1, 1) and let $f(x)$ be defined by

$$f(x) = \begin{cases} -x^2 forx & \text{in} \quad (-1,0), \\ x^3 forx & \text{in} \quad (0,1). \end{cases}$$

Show that the weak derivative $f'(x)$ exists or doesn't exist.

B3. (i) Define what a standard mollifier in $\mathbb{R}^N$ is.

(ii) Given Ω to be a domain in $\mathbb{R}^N$ and let K be a compact subset of Ω. Further, let $h = (h_1, ..., h_N)$ be a translation in $\mathbb{R}^N$ and define

$Thf(x) = f(x+h), x \in K \forall f \in L^P_{loc}(\Omega), P \geq 1.$

By using the f_ϵ, the mollification of f over K, prove that

$$\lim_{|h| \to 0} \|T_h f - f\|_{L^p(K)} = 0.$$

(Hint: Use the fact that $f\epsilon \to f$ in $L^P_{loc}(\Omega)$ as $\epsilon \to 0^+$)

B4. Write down an explicit formula for a solution of the Cauchy Heat problem

$$\begin{cases} u_t - \Delta u + cu = f & \text{in} \quad \mathbb{R}^N x(0,\infty), \\ u(0) = g & \text{on} \quad \mathbb{R}^N. \end{cases}$$

B5. (i) Consider the partial differential operator

$L = \frac{\partial^2}{\partial t^2} - \frac{c^2\partial^2}{\partial x^2}$,

Prove that if $Lu = Lv - 0$ for $x \notin (a, b)$ and $u(a) = v(a) = 0, u(b) = v(b) = 0$, then $\frac{d}{dt}\int_a^b \frac{1}{2}(u_t v_t + c^2 u_x v_x)dx = 0$.

(ii) Solve the 1-D Wave Cauchy Problem

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & \text{in} \quad \mathbb{R}x(0, \infty) \\ u(x, 0) = e^x, u_t(x, 0) = & \text{in} \quad \sin x on \mathbb{R} \end{cases}$$