# Ph.D. Qualifying Examination D
# Differential Equations
# January 2016

**Instructions**: In this three-hour examination, Part A and Part B carry equal weight in determining your overall performance. Please use separate blue books for Part A and Part B.

***Answer all 4 questions in Part A and 4/5 questions in Part B.***

# PART A

A1. Let $f(t,x) \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and that $f$ satisfies a local Lipschitz condition in $x$. Assume $f(t,0) = 0$. If $x(t)$ is a solution of the equation $\dot{x} = f(t,x)$ such that $x(0) \neq 0$, show that $x(t) \neq 0$ for any $t \in \mathbb{R}$.

A2. Show that for some $\mu \neq 0$, the system

$$\begin{aligned} \dot{x} &= \mu x - y + xy - xy^2 - x^3 \\ \dot{y} &= x + \mu y - x^2 - y^3 \end{aligned}$$

has a nonconstant periodic orbit.

A3. Consider the system $\dot{x} = Ax$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Let $x(t)$ be a solution of the above system. We say $x(t)$ grows linearly if $\lim_{t\to\infty} \frac{|x(t)|}{t} = c > 0$, and grows superlinearly (faster than linearly) if $\lim_{t\to\infty} \frac{|x(t)|}{t} = \infty$. Find *all* initial conditions $x(0)$ such that the respective solutions $x(t)$

(a) Are bounded.

(b) Grow linearly.

(c) Grow superlinearly.

A4. Consider

$$\ddot{x} + \alpha\dot{x} + g(x) = 0 \tag{1}$$

with $\alpha > 0$, $g$ a $C^1$ function with $xg(x) > 0$ for $x \neq 0$, $\int_0^{-\infty} g(x)\,dx = \infty$, and $\int_0^{\infty} g(x)\,dx = c < \infty$. Prove that every bounded solution $x(t)$ of equation (1) satisfies

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \dot{x}(t) = 0$$

and that every solution $x(t)$ is bounded on $t \in (0,\infty)$.

# $\mathcal{PART\ B}$

B1. (i) Let $\Omega$ be a domain in $\mathbb{R}^N$, $N \geq 1$, given $u \in L^p(\Omega)$, $p \geq 1$, define $\frac{\partial u}{\partial xi}$ $1 \leq i \leq N$, the weak derivatives of first order for u and hence, the Sobolev Space $W^{1,p}(\Omega)$.

(ii) For $u(x) = \frac{1}{|x|^\alpha}$, $\alpha > 0$, what are the values for $\alpha$ and p in terms of N, would u belongs to $W^{1,p}(\Omega)$? If you wish, you may identify $\Omega$ with B(0,1), the unit ball in $\mathbb{R}^N$.

B2. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 1$ which has a $C^1$ boundary $\partial\Omega$ and consider the Poission Dirichlet problem,

$$\begin{cases} \Delta u = f & \text{in} \quad \Omega, f \in L^p(\Omega), p \geq 1 \\ u = 0 & \text{on } \partial\Omega \quad \text{in the sense of trace.} \end{cases}$$

Suppose we formulate the solution by seeking $u \in W_0^{2,p}(\Omega)$ such that $-\Delta u = f$, criticize this formulation and if you don't agree with the formulation, how would you reformulate it? Explain your answer.

B3. Consider the non-homogeneous Poission Dirichlet problem,

$$\begin{cases} \Delta u = f & \text{in } B(0;R), \quad \text{the ball with radius } R \text{ in } \mathbb{R}^N \text{with} N \geq 2, \\ u = g & on\ \partial B(0;R) \end{cases}$$

Prove that, for its solution u, we have the mean-value formula,

$$u(0) = \frac{1}{\omega_N R^{N-1}} \int_{\partial B(0,R)} g(\xi) dS_\xi + \frac{1}{(N-2)\omega_N} \int_{B(0,R)} (\frac{1}{R^{N-2}} - \frac{1}{|\xi|^{N-2}}) f(\xi) d\xi.$$

where $\frac{1}{\omega_N R^{N-1}} \int_{\partial B(0,R)} g(\xi) dS_\xi$ being the mean value of g over the sphere $\partial B(0,R)$, and $\omega_N$ is the area of the spherical surface of radius 1 in $\mathbb{R}^N$.

B4. Consider the PDE of the Neuman type,

$$\begin{cases} \Delta u = 1 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

Where n is the exterior unit normal vector at $\partial\Omega$, prove that this problem has no solution.

B5. (i) Solve the 1-D Wave Cauchy Problem

$$\begin{cases} u_{tt} - 4u_{xx} = 0, -\infty < x < \infty, t > 0, \\ u(x,0) = \sin x, -\infty < x < \infty, \\ u_t(x,0) = \cos x, -\infty < x < \infty. \end{cases}$$

(ii) Consider now the wave equation in the $\mathbb{R}^N$ setting, $N \geq 2$,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \quad \text{in} \quad \Omega, t > 0, c > 0, \\ u(x,0) = g(x), u_t(x,0) = h(x), x \in \Omega, \\ u(x,t) = f(x), x \in \partial\Omega. \end{cases}$$

Prove that u is unique (Hint: Let u and v both be solutions and set w=u-v, show that w=0).